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An Algebraic Structure of Orthogonal Wavelet Space [1]

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In this paper we study the algebraic structure of the space of compactly supported orthonormal wavelets over real numbers. Based on the parameterization of wavelet space, one can define a parameter mapping from the wavelet space of rank 2 (or 2-band, scale factor of 2) and genus g to the $(g-1)$ dimensional real torus (the products of unit circles). By the uniqueness and exactness of factorization, this mapping is well defined and one-to-one. Thus we can equip the rank 2 orthogonal wavelet space with an algebraic structure of the torus. Because of the degenerate phenomenon of the paraunitary matrix, the parameterization map is not onto. However, there exists an onto mapping from the torus to the closure of the wavelet space. And with such mapping, a more complete parameterization is obtained. By utilizing the factorization theory, we present a fast implementation of discrete wavelet transform (DWT). In general, the computational complexity of a rank m orthogonal DWT is $O(m^2 g)$. In this paper we start with a given scaling filter and construct additional $(m-1)$ wavelet filters so that the DWT can be implemented in $O(mg)$. With a fixed scaling filter, the approximation order, the orthogonality, and the smoothness remain unchanged; thus our fast DWT implementation is quite general.

Key Words: wavelets; parameterization; fast implementation

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1. INTRODUCTION

Wavelet analysis (see [2, 3, 9, 17, 25, 27, 31, 38, 39], etc.) has been proven to be a very powerful tool in harmonic analysis, neural networks, numerical analysis, and signal processing, especially in the area of image compression [28, 32, 33, 40] and noise removal [7, 11, 21]. The theoretical work of orthogonal wavelets was done in the late 1980s [8, 24, 25] and the framework of biorthogonal wavelets was established in the early 1990s [4, 6, 37]. Wavelet theory is closely connected with subband coding, and it provides a functional space structure for subband coding, often leading to better understanding and tremendous improvement.

The invention of the polyphase decomposition [1] is one of the reasons why multirate filter banks processing [34] became practically attractive. It enables the designer to perform all the computations at the lowest rate permissible within the given context, and reduces the speed requirements on the processors. It is valuable not only in practical design and actual implementation of filter banks, but also in theoretical study. Actually with the polyphase decomposition, Vaidyanathan and his colleagues [35] derive factorizations of paraunitary matrices and apply such factorizations to design quadrature mirror filter (QMF) banks for digital signal processing problems. Pollen, Heller, Resnikoff, and Wells, Jr. [16, 26, 27] use the polyphase decomposition to develop a parameterization theory of compactly supported orthonormal wavelets.

In this paper, we will discuss the algebraic structure of the space of compactly supported orthonormal wavelets over real numbers. The factorization work mentioned above suggests a mapping from the orthogonal wavelet space to the space of products of unit spheres. But without the uniqueness of the factorization, the mapping cannot be well defined. The uniqueness result for rank $m = 2$ (or 2-band, scale factor of 2) was first proved by Pollen [26] and one can define a mapping from the orthogonal wavelet space of rank 2 to products of unit circles. However, this mapping does not provide a quantitative relationship between the size of a wavelet system (or more precisely, the genus of a wavelet matrix) and the number of unit circles in the product. Here we present a new result of the wavelet factorization, the *exactness* of factorization, which says that the number of unit circles in the product will be exactly the genus minus 1. Then with the uniqueness and exactness of wavelet factorization, one can define a one-to-one mapping from the orthogonal wavelet space of rank 2 and genus g (which means the scaling filter has length $2g$) to the $(g - 1)$ dimensional real torus

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}.$$

Because of the degenerate phenomenon of the paraunitary matrix, this mapping is not onto. Considering even genus and odd genus orthogonal wavelets as two different classes and taking the union in each class up to some genus, that is, adding wavelets with smaller genus but with the same parity to build up a larger space (which is called the closure in this paper), then there exists an onto mapping from the torus to the closure with which an inclusion diagram commutes (see Theorem 3.4). When $m > 2$, there is no well-defined mapping from the orthogonal rank m wavelet space to the products of unit spheres. However, one can still set up a parameter mapping from the real torus (products of unit spheres) to the orthogonal wavelet space.

In [35], it is shown that a paraunitary matrix of (McMillan) degree $N-1$ can be factored as the product of $N-1$ primitive paraunitary matrices and a constant matrix. This result is different from our exactness result here, because the McMillan degree is greater than or equal to $g-1$, where g is the genus. An example is given in Section 3.1 which illustrates that the McMillan degree (which is 2 in the example) can be greater than $g-1$ (which is 1). Also in [35], the authors proved the uniqueness of factorization for lossless column vector; for a paraunitary matrix (of rank m), however, as discussed in [35], the factorization is not unique in general.

In [10], a wavelet matrix (orthogonal or biorthogonal) is decomposed into a finite sequence of lifting steps, which asymptotically reduces the computational complexity of the transform by a factor of two. Their factorization form is different from ours in the sense that we restrict the prime factors (which are primitive paraunitary matrices in this paper) to be orthogonal, so that the product of prime factors (plus an orthogonal Haar wavelet matrix) will always give an orthogonal wavelet matrix.

In [14, 17, 18], a factorization-based loop group is presented. By extending 2×2 unitary operators on the fundamental domain, they constructed a family of unitary operators that commute with the translations by 2. Such a family together with the translation operator T_1 (translation by 1) generates all orthogonal wavelet matrices of rank 2.

From the factorization theory, we present a fast implementation of the discrete wavelet transform (DWT). It gives an optimal solution to the problem of designing orthogonal wavelet filters from a fixed scaling filter, where the optimality is measured in the sense of DWT computational complexity.

This paper is organized as follows. In Section 2 we give a brief review of previous work on paraunitary matrices and orthonormal wavelets. The exactness and uniqueness of wavelet factorization for rank $m = 2$ is discussed in Section 3. In particular we present a new proof of Pollen's uniqueness result based on our exactness result. This leads to a particular algebraic topological structure for the space of compactly supported orthonormal wavelets over real numbers. More specifically, there exists a well-defined one-to-one mapping from the orthogonal wavelet space of rank 2 and genus g to the $(g-1)$ dimensional real torus. In Section 4 we study a fast wavelet transform implementation based on factorization. We conclude the paper in Section 5.

2. A REVIEW OF PREVIOUS WORK

Most of the material in this section is from [1, 16, 20, 26, 27, 29, 34–36]. We refer to these papers and books for more details.

2.1. Laurent Polynomial and Polyphase Decomposition

For a given sequence $a = \{a_k,\ k \in \mathbf{Z}\}$ which has only finite nonzero elements, the *Laurent polynomial* $a(z)$ of a is defined by

$$a(z) := \sum_{k\in\mathbf{Z}} a_k z^{-k} = \sum_{k=k_1}^{k_2} a_k z^{-k},$$

where k_1 and k_2 are the smallest and largest indices that a_k is nonzero, respectively. In signal processing, a will be called a *finite impulse response* (FIR) *filter* and $a(z)$ is the *z-transform* of a.

Now consider a matrix $A = (a_{i,j})$ consisting of m rows of vectors of the form

$$A = \begin{pmatrix} \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ & & \vdots & \vdots & & \\ \cdots & a_{m-1,-1} & a_{m-1,0} & a_{m-1,1} & a_{m-1,2} & \cdots \end{pmatrix},$$

where only a finite number of entries $a_{i,j}$ are nonzero, and $m \in \mathbf{N}$, $m \geq 2$. Define submatrices A_k of size $m \times m$ of A in the following manner,

$$A_k = (a_{i,km+j}), \qquad i = 0, \ldots, m-1, \; j = 0, \ldots, m-1,$$

for $k \in \mathbf{Z}$. In other words, A is expressed in terms of block matrices in the form

$$A = (\ldots, A_{-1}, A_0, A_1, \ldots),$$

where, for instance,

$$A_0 = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m-1} \\ \vdots & & & \vdots \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,m-1} \end{pmatrix}.$$

From the matrix A, we construct the formal power series

$$A(z) := \sum_{k \in \mathbf{Z}} A_k z^{-k} = \sum_{k=k_1}^{k_2} A_k z^{-k}, \tag{1}$$

where k_1 and k_2 are the smallest and largest indices that $A_k \neq 0$, respectively. We call $A(z)$ the *Laurent series* of the matrix A. We can equally well write $A(z)$ as an $m \times m$ matrix,

$$A(z) = \begin{pmatrix} \sum_k a_{0,km} z^{-k} & \cdots & \sum_k a_{0,km+m-1} z^{-k} \\ \vdots & & \vdots \\ \cdots & \sum_k a_{i,km+j} z^{-k} & \cdots \\ \vdots & & \vdots \\ \sum_k a_{m-1,km} z^{-k} & \cdots & \sum_k a_{m-1,km+m-1} z^{-k} \end{pmatrix},$$

which we will refer to as the *polyphase decomposition* of A. For the case of $m = 2$, we find

$$A(z) = \begin{pmatrix} \cdots + a_{0,-2}z + a_{0,0} + a_{0,2}z^{-1} + \cdots, & \cdots + a_{0,-1}z + a_{0,1} + a_{0,3}z^{-1} + \cdots \\ \cdots + a_{1,-2}z + a_{1,0} + a_{1,2}z^{-1} + \cdots, & \cdots + a_{1,-1}z + a_{1,1} + a_{1,3}z^{-1} + \cdots \end{pmatrix}$$

and we see that the even and odd coefficients along the rows are blocks in the left and right columns, respectively.

Let

$$g := k_2 - k_1 + 1$$

be the number of terms in the summation (1) and call g the *genus* of the Laurent series $A(z)$ and the matrix A. Thus A has a size of $m \times mg$.

Finally we define the *adjoint* $\tilde{A}(z)$ of the Laurent series $A(z)$ by

$$\tilde{A}(z) := A^*(z^{-1}) := \sum_{k=k_1}^{k_2} A_k^* z^k = \sum_{k=-k_2}^{-k_1} A_{-k}^* z^{-k},$$

where $A_k^* := \bar{A}_k^t$ is the Hermitian adjoint of the $m \times m$ matrix A_k. When A_k is a real matrix, $A_k^* = A_k^t$.

2.2. Paraunitary Matrices

For an $m \times mg$ matrix A, we say that A is *paraunitary* if its Laurent series $A(z)$ satisfies

$$A(z)\tilde{A}(z) = cI_m, \tag{2}$$

where $c \neq 0$ is a constant and I_m is the $m \times m$ identity matrix. This generalizes the classical notion of unitary matrix (the case where $g = 1$ and $c = 1$). In signal processing, one can design perfect reconstruction (PR) FIR filter banks with particular properties from a paraunitary matrix A.

Let v be a unit column vector; that is, $v^* v = 1$. Define the Laurent matrix

$$V(z) := I_m - vv^* + vv^* z^{-1} \tag{3}$$

and let V be the corresponding $m \times 2m$ matrix. It is easy to verify that V is a paraunitary matrix. We will say that a paraunitary matrix V of the form (3) is *primitive*.

Note that a primitive paraunitary matrix has the constant $c = 1$ in (2). And any primitive paraunitary matrix has determinant z^{-1}. The proof can be found, for example, in [35].

LEMMA 2.1. *If V is a primitive paraunitary matrix, then* $\det(V(z)) = z^{-1}$.

A fundamental result is that an arbitrary paraunitary matrix can be factored into products of primitive paraunitary matrices. We define the *paraunitary product* of two paraunitary matrices A and B to be given by

$$AB = C$$

if

$$A(z)B(z) = C(z), \tag{4}$$

where $A(z)$, $B(z)$, and $C(z)$ are the Laurent series of A, B, and C, and the product in (4) is simply the matrix product.

THEOREM 2.1 (Paraunitary factorization theorem). *Let A be an $m \times mg$ paraunitary matrix, then there must exist primitive paraunitary matrices $V_1, \ldots, V_d$ and a square $m \times m$ paraunitary matrix U such that*

$$A = V_1 V_2 \cdots V_d U,$$

where $d \in \mathbf{N}$, $d \geq g - 1$, and $U = z^{-k_1} A(1)$, where $A(z)$ is the Laurent series of A, and k_1 is the smallest index that $A_k \neq 0$ in (1).

See [35] for a complete proof.

2.3. Orthogonal Wavelet Matrices

An $m \times mg$ matrix $A = (a_{i,j})$ is said to be an *orthogonal wavelet matrix* of *rank m* if

$$A(z)\tilde{A}(z) = m I_m \tag{5}$$

and

$$\sum_j a_{i,j} = \begin{cases} m & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq m-1. \end{cases} \tag{6}$$

It is easy to verify that an orthogonal wavelet matrix with m rows has rank m in the classical sense.

Note that in the theory of wavelet analysis, we will systematically employ the additional linear constraint (6) in addition to the paraunitary condition (5). This is one of the main differences between wavelet systems and PR FIR filter banks.

Comparison of coefficients of corresponding powers of z in (5) yields quadratic orthogonality relations for the rows of A,

$$\sum_{j \in \mathbf{Z}} \bar{a}_{i_1, k_1 m + j} a_{i_2, k_2 m + j} = m \delta_{i_1, i_2} \delta_{k_1, k_2}, \tag{7}$$

where δ is defined by

$$\delta_{i_1, i_2} := \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to (5) and (6) or equivalently (7) and (6) as the *quadratic* and *linear* conditions defining an orthogonal wavelet matrix, respectively.

From the theory of orthogonal wavelets [8, 9, 27, 30], one can define a *scaling function* $\phi(x)$ and $m - 1$ *wavelet functions* $\psi_1(x), \ldots, \psi_{m-1}(x)$, all are compactly supported L^2 functions, by

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} M_0(m^{-j}\xi),$$

where $\hat{\phi}(\xi)$ is the Fourier transform of $\phi(x)$

$$\hat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{-i\xi x}\, dx$$

and $M_0(\xi)$ is the discrete Fourier transform of $\{a_{0,k},\ k \in \mathbf{Z}\}$

$$M_0(\xi) := \frac{1}{m} \sum_k a_{0,k} e^{-ik\xi}$$

and

$$\psi_i(x) := \sum_k a_{i,k} \phi(mx - k), \qquad i = 1, \ldots, m-1.$$

For $j, k \in \mathbf{Z}$ we define the *rescaled* and *translated* scaling and wavelet functions by

$$\phi_{j,k}(x) := m^{j/2} \phi(m^j x - k)$$
$$(\psi_i)_{j,k}(x) := m^{j/2} \psi_i(m^j x - k), \qquad i = 1, \ldots, m-1.$$

With the quadratic condition (5) and linear condition (6), the wavelet functions $\psi_1(x), \ldots, \psi_{m-1}(x)$ will generate a tight frame [22].

THEOREM 2.2 (Lawton). *Let A be an orthogonal wavelet matrix of rank m and genus g, and let $\psi_1(x), \ldots, \psi_{m-1}(x)$ and their rescaled and translated version be defined as above. Then, for all $f \in L^2(\mathbf{R})$, there exists an L^2 convergent expansion*

$$f(x) = \sum_{i=1}^{m-1} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (d_i)_{j,k} (\psi_i)_{j,k}(x),$$

where the coefficients are given by

$$(d_i)_{j,k} = \int f(x) (\psi_i)_{j,k}(x)\, dx.$$

The necessary and sufficient conditions that $(\psi_i)_{j,k}(x)$ constitute an orthonormal basis for $L^2(\mathbf{R})$ are given by Cohen [5] and Lawton [23]. One needs to either find a good compact set congruent to $[-\pi, \pi]$ modulo 2π or check whether 1 is a nondegenerate eigenvalue of a multiresolution operator to ensure orthogonality. We refer to [5, 23] for more details.

Now we return to orthogonal wavelet matrices. The set of orthogonal wavelet matrices with genus equal to 1 play a special role in the theory of orthogonal wavelets. We shall call them *orthogonal Haar wavelet matrices*. The set of orthogonal Haar wavelet matrices of rank m is a homogeneous space which is isomorphic to the Lie group U_{m-1} of unitary $(m-1) \times (m-1)$ matrices, and there is a distinguished Haar matrix which corresponds to the identity element of the group U_{m-1}, which will be called the *canonical Haar matrix*.

LEMMA 2.2. *An $m \times m$ matrix H is an orthogonal Haar wavelet matrix if and only if*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \mathbf{H},$$

where $U \in U_{m-1}$ is a unitary matrix, that is, $U^*U = I_{m-1}$, and $\mathbf{H}$ *is the* canonical Haar matrix *of rank m, which is defined by*

$$\mathbf{H} := \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ -\sqrt{m-1} & \sqrt{\frac{1}{m-1}} & \cdots & \cdots & \cdots & \cdots & \sqrt{\frac{1}{m-1}} \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\sqrt{\frac{im}{i+1}} & \sqrt{\frac{m}{i^2+i}} & \cdots & \sqrt{\frac{m}{i^2+i}} \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{\frac{m}{2}} & \sqrt{\frac{m}{2}} \end{pmatrix},$$

where $i = m-1, \ldots, 2, 1$.

The proof of the above lemma can be found, for example, in [16, 27]. Let A be an orthogonal wavelet matrix and let $A(z)$ be its Laurent series. Define the *characteristic Haar matrix* $\chi(A)$ of the wavelet matrix A by

$$\chi(A) := A(1).$$

It can be easily checked that χ is a well-defined mapping from orthogonal wavelet matrices of rank m to orthogonal Haar wavelet matrices of rank m. When $m = 2$, we have

$$\chi(A) = \begin{pmatrix} \cdots + a_{0,-2} + a_{0,0} + a_{0,2} + \cdots, & \cdots + a_{0,-1} + a_{0,1} + a_{0,3} + \cdots \\ \cdots + a_{1,-2} + a_{1,0} + a_{1,2} + \cdots, & \cdots + a_{1,-1} + a_{1,1} + a_{1,3} + \cdots \end{pmatrix}.$$

With Lemma 2.2, the following theorem is a simple consequence of Theorem 2.1.

THEOREM 2.3 (First orthogonal factorization theorem). *If A is an orthogonal wavelet matrix of rank m and genus g, then there exist primitive paraunitary matrices $V_1, \ldots, V_d$ such that*

$$A = z^{-k_1} V_1 V_2 \cdots V_d H, \tag{8}$$

where $d \in \mathbf{N}$, $d \geq g-1$, $H = \chi(A)$ is the characteristic Haar matrix of A, and k_1 is the smallest index that $A_k \neq 0$ in the Laurent series (1).

The number d in (8) can by determined in the following way. From (5) it follows that $\det(A(z)) = cz^{-b}$ for some constant $c \neq 0$ and integer b. Now taking the determinant on both sides of (8) and using the result of Lemma 2.1, we get

$$cz^{-b} = z^{-mk_1} z^{-d} \det(H).$$

Note that

$$\det(H) = \det(A(1)) = c$$

so

$$b = mk_1 + d.$$

Thus

$$d = b - mk_1.$$

In particular, if $k_1 = 0$, then $d = b$.

For two orthogonal wavelet matrices A_1 and A_2, whose characteristic Haar matrix are the same, we can define the *Pollen product* by the formula

$$A_1 \diamond_H A_2 := A$$

if their Laurent series satisfy

$$A(z) = A_1(z) H^{-1} A_2(z),$$

where

$$H = \chi(A_1) = \chi(A_2).$$

It is easy to check that the characteristic Haar matrix of A is also H. Thus the set of orthogonal wavelet matrices with the same characteristic Haar matrix is a group with the (noncommutative) Pollen product. Thus we may restate Theorem 2.3 in the language of Pollen products.

THEOREM 2.4 (Second orthogonal factorization theorem). *If A is an orthogonal wavelet matrix A of rank m and genus g, then there exist orthogonal wavelet matrices $A_1, \ldots, A_d$ such that*

$$A = z^{-k_1} A_1 \diamond_H A_2 \diamond_H \cdots \diamond_H A_d,$$

where $H = \chi(A)$ is the characteristic Haar matrix of A, and

$$A_j(z) = V_j(z) H = \left(I_m - v_j v_j{}^* + v_j v_j{}^* z^{-1}\right) H, \qquad j = 1, \ldots, d,$$

where for each v_j, $v_j{}^* v_j = 1$.

3. THE CASE OF RANK $M = 2$

From the factorization theorems in Section 2, we know that an orthogonal wavelet matrix can be written as a product of some primitive paraunitary matrices and an orthogonal Haar wavelet matrix. Thus it suggests a mapping from the space of orthogonal wavelet matrices to the space of products of primitive paraunitary matrices (which is, actually, unit spheres, by definition) and orthogonal Haar wavelet matrices. However, such a mapping cannot be well defined without knowing

1. The factorization form is unique. From an orthogonal wavelet matrix, there is only one set of primitive paraunitary matrices and an orthogonal Haar wavelet matrix which correspond to the factorization form.
2. The number of primitive paraunitary matrices, d, can be quantitatively determined by the genus g. Then one can set up a mapping from the subspace of orthogonal wavelet matrices of genus g to a product of a torus and an orthogonal Haar wavelet matrix.

We will refer to these two requirements as the uniqueness and exactness of factorization, respectively. In this section, we will present the exactness of factorization for rank $m = 2$ and also give a new proof of Pollen's uniqueness result for rank $m = 2$. It turns out only for $m = 2$ that the uniqueness and exactness of factorization are valid. We will mainly study the case of $m = 2$ in this section, but will also discuss the case of $m > 2$ as well.

3.1. Uniqueness and Exactness of Factorization

In the literature, a two-band (or two-channel) filter bank has been referred to as the *quadrature mirror filter* (QMF) bank [12, 13, 19]. In wavelet analysis, the most common setting for the rank m is $m = 2$, in which we will have one scaling function $\phi(x)$ and one wavelet function $\psi(x)$. Thus when decomposing a function, there will be exactly two parts in the wavelet representation, the average part of $\phi(x)$ and the difference part of $\psi(x)$.

In $\mathbf{R}^2$ a unit vector can be represented by a point $(\cos\theta, \sin\theta)'$ on the unit circle S^1, and its primitive paraunitary matrix

$$\begin{aligned} V_\theta(z) &= I_2 - \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) z^{-1} \\ &= I_2 - \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} z^{-1} \\ &= \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} z^{-1} \\ &= S(\theta) + C(\theta) z^{-1}, \end{aligned}$$

where

$$\begin{aligned} S(\theta) &:= \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} \\ C(\theta) &:= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \end{aligned}$$

Note that $S(\theta)$ and $C(\theta)$ are related by

$$S(\theta) = C(\theta + \pi/2)$$

and

$$C(\theta) = S(\theta + \pi/2).$$

Because of this special form of primitive paraunitary matrix, the orthogonal wavelet matrix over real numbers will have exactly $g - 1$ primitive paraunitary matrix factors in the factorization.

THEOREM 3.1 (Exactness of factorization). *If A is a real orthogonal wavelet matrix of rank 2 and genus g, then there exist exactly $g - 1$ primitive paraunitary matrices $V_1, \ldots, V_{g-1}$ such that*

$$A = z^{-k_1} V_1 V_2 \cdots V_{g-1} H, \tag{9}$$

where k_1 is the smallest index that $A_k \neq 0$ in the Laurent series (1), *H is the characteristic Haar matrix of A.*

Proof. By Theorem 2.3, there exist d primitive paraunitary matrices $V_1, \ldots, V_d$ such that

$$\begin{aligned} A(z) &= z^{-k_1} V_1(z) \cdots V_d(z) H \\ &= z^{-k_1} \left(S(\theta_1) + C(\theta_1) z^{-1}\right) \cdots \left(S(\theta_d) + C(\theta_d) z^{-1}\right) H \\ &= z^{-k_1} \left(S(\theta_1) \cdots S(\theta_d) + \cdots + C(\theta_1) \cdots C(\theta_d) z^{-d}\right) H. \end{aligned}$$

To prove $d = g - 1$, it suffices to show the leading coefficient

$$C(\theta_1) C(\theta_2) \cdots C(\theta_d) \neq 0.$$

If not, one could have

$$C(\theta_1) C(\theta_2) \cdots C(\theta_d) = 0.$$

Since

$$\begin{aligned} &C(\theta_1) C(\theta_2) \cdots C(\theta_d) \\ &= \begin{pmatrix} \cos^2\theta_1 & \cos\theta_1 \sin\theta_1 \\ \cos\theta_1 \sin\theta_1 & \sin^2\theta_1 \end{pmatrix} \cdots \begin{pmatrix} \cos^2\theta_d & \cos\theta_d \sin\theta_d \\ \cos\theta_d \sin\theta_d & \sin^2\theta_d \end{pmatrix} \\ &= \left(\prod_{j=1}^{d-1} \cos(\theta_j - \theta_{j+1}) \right) \begin{pmatrix} \cos\theta_1 \cos\theta_d & \cos\theta_1 \sin\theta_d \\ \sin\theta_1 \cos\theta_d & \sin\theta_1 \sin\theta_d \end{pmatrix} \end{aligned}$$

it follows that for some $j_0 \in \{1, \ldots, d-1\}$, $\cos(\theta_{j_0} - \theta_{j_0+1}) = 0$. Thus

$$\begin{aligned} &S(\theta_1) S(\theta_2) \cdots S(\theta_d) \\ &= C(\theta_1 + \pi/2) \cdots C(\theta_d + \pi/2) \\ &= \left(\prod_{j=1}^{d-1} \cos(\theta_j - \theta_{j+1}) \right) \begin{pmatrix} \sin\theta_1 \sin\theta_d & -\sin\theta_1 \cos\theta_d \\ -\cos\theta_1 \sin\theta_d & \cos\theta_1 \cos\theta_d \end{pmatrix} \\ &= 0 \end{aligned}$$

which is impossible, since

$$S(\theta_1) S(\theta_2) \cdots S(\theta_d) = A_{k_1} \neq 0.$$

So d must be equal to $g - 1$. ■

The following uniqueness theorem was first derived by Pollen [26].

THEOREM 3.2 (Uniqueness of factorization). *For an orthogonal wavelet matrix A of rank* 2 *and genus g, the factorization is unique. If*

$$A = z^{-k_1} V_1 V_2 \cdots V_d H = z^{-k_1} W_1 W_2 \cdots W_d G,$$

where V_j, W_j are primitive paraunitary matrices, $j = 1, \ldots, d$, H and G are two square constant matrices, then $V_j = W_j$, $j = 1, \ldots, g-1$, and $H = G$.

Note that from Section 2, we know that if $\det(A(z)) = cz^{-b}$, then

$$d = b - mk_1;$$

thus the number of factors, d, will be invariant if $A(z)$ has two or more different factorizations, which actually cannot happen when $m = 2$.

The uniqueness of factorization can actually be derived from the exactness result. The reason is that if

$$A = z^{-k_1} V_1 V_2 \cdots V_{g-1} H = z^{-k_1} W_1 W_2 \cdots W_{g-1} G$$

then evaluating at $z = 1$ will imply $H = G$, and

$$V_1 V_2 \cdots V_{g-1} (W_{g-1})^{-1} \cdots (W_2)^{-1} (W_1)^{-1} = I_2.$$

Note that W_j is a primitive paraunitary matrix

$$W_j = S(\eta_j) + C(\eta_j) z^{-1}$$

so

$$\begin{aligned}
(W_j)^{-1} &= (W_j)^* \\
&= S(\eta_j) + C(\eta_j) z \\
&= z\big(C(\eta_j) + S(\eta_j) z^{-1}\big) \\
&= z\big(S(\eta_j + \pi/2) + C(\eta_j + \pi/2) z^{-1}\big) \\
&= z U_j,
\end{aligned}$$

where

$$U_j = S(\eta_j + \pi/2) + C(\eta_j + \pi/2) z^{-1}.$$

Thus we have

$$V_1 V_2 \cdots V_{g-1} U_{g-1} \cdots U_2 U_1 = z^{-g+1}.$$

Since z^{-g+1} has genus 1, and $d = g - 1 = 0$, by the exactness of factorization, we must have

$$V_j U_j = z^{-1} I_2, \qquad j = 1, \ldots, g-1,$$

that is,

$$V_j = W_j$$

which implies the uniqueness of factorization.

When $m > 2$, the uniqueness result or the exactness do not hold in general. Here is a counterexample:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z^{-1} &= \big(I_3 - e_2 e_2{}^* + e_2 e_2{}^* z^{-1}\big)\big(I_3 - e_3 e_3{}^* + e_3 e_3{}^* z^{-1}\big) \\
&= \big(I_3 - v_2 v_2{}^* + v_2 v_2{}^* z^{-1}\big)\big(I_3 - v_3 v_3{}^* + v_3 v_3{}^* z^{-1}\big),
\end{aligned}$$

where

$$e_2 = (0, 1, 0)', \qquad e_3 = (0, 0, 1)'$$

and

$$v_2 = \left(0, \sqrt{2}/2, -\sqrt{2}/2\right)', \qquad v_3 = \left(0, \sqrt{2}/2, \sqrt{2}/2\right)'.$$

In this example, we have two different factorizations and $d = 2 > g - 1 = 1$. This example also illustrates the difference between the exactness result Theorem 3.1 and the one in [35], since its McMillan degree is equal to 2, which is equal to d, but greater than $g - 1$. Even when $m = 2$, if A is a complex matrix, the number of factors, d, still can be greater than $g - 1$, even though $A(z)$ has a unique factorization. One can choose two complex unit vectors $v_1, v_2 \in \mathbf{C}^2$ such that they are orthogonal to each other but

$$(I_2 - v_1 v_1^*)(I_2 - v_2 v_2^*) \neq 0,$$

then the product of their primitive paraunitary matrices will have genus $g = 2$, but $d = 2 > g - 1$.

Another remark is that in the factorization, the exponent k_1 is set to be the smallest index that $A_k \neq 0$ in the Laurent series $A(z)$. If we drop this requirement on k_1, then $A(z)$ will have different factorization because of the degenerate phenomenon

$$V_{\theta_1}(z) V_{\theta_2}(z) = z^{-1} I_2$$

if $\theta_1 - \theta_2 = k\pi + \pi/2$ for some integer k. Thus the factorization given by Theorems 2.1 and 2.3 is minimal in the sense that the number of factors is minimal.

3.2. Algebraic Structure of Orthogonal Wavelet Space

Combining Theorems 3.1 and 3.2, if A is a real orthogonal wavelet matrix of rank 2 and genus g, there exist exactly and uniquely $g - 1$ primitive paraunitary matrices $V_1, \ldots, V_{g-1}$ such that

$$A(z) = z^{-k_1} V_1 \cdots V_{g-1} H, \tag{10}$$

where $H = A(1)$ is the characteristic Haar matrix. Since $m = 2$, we have

$$H = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{or} \quad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus for any real orthogonal wavelet matrix of rank 2 and genus g, there exists a unique set $\{k_1, V_1, V_2, \ldots, V_{g-1}, H\}$ such that (10) holds. By definition,

$$V_j = I_m - v_j v_j^* + v_j v_j^* z^{-1},$$

where v_j is a unit vector, $v_j^* v_j = 1$, $j = 1, 2, \ldots, g - 1$. Then $V_1 V_2 \cdots V_{g-1}$ can be identified as a point on the $(g - 1)$ dimensional real torus

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}.$$

So the uniqueness of factorization sets up a well-defined mapping from the space of orthogonal wavelets to the space of real tori (with additional parameter k_1 and Haar matrix H, which will be discussed next). And the exactness of factorization guarantees the image set of the subspace of genus g orthogonal wavelets will be inside the $(g-1)$ dimensional real torus.

Since the multiplication by z^{-k} is just a shift of index j in $a_{i,j}$, we will discard the difference between two orthogonal wavelet matrices if one is another multiplied by z^{-k} for some integer k. Note that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we assume

$$A(z) = \begin{pmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{pmatrix},$$

then

$$A(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h_o(z) & h_e(z) \\ g_o(z) & g_e(z) \end{pmatrix},$$

which means that a multiplication by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a flip on the components (even and odd coefficients of A at each row) and does not change the structure of $A(z)$. Without loss of generality, we will always assume that

$$H = A(1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

or equivalently

$$\sum_{j \in \mathbf{Z}} a_{1,2j} = -1$$
$$\sum_{j \in \mathbf{Z}} a_{1,2j+1} = 1.$$

Then a real orthogonal wavelet matrix A of rank 2 and genus g can be uniquely characterized by $g-1$ primitive paraunitary matrices, or $g-1$ unit vectors, which is an element of the $(g-1)$ dimensional real torus. So the uniqueness and exactness of factorization builds up a bridge between the space of compactly supported orthonormal wavelets over real numbers and a torus, which is the product of unit circles.

Let $WM(2, g)$ be the collection of all real orthogonal wavelet matrices of rank 2 and genus g, and $WM(2)$ be the collection of all real orthogonal wavelet matrices of rank 2, then

$$WM(2) = \bigcup_{1 \le g < \infty} WM(2, g).$$

And let T^n denote the n-dimensional real torus

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}.$$

As we know, the image set of $WM(2, g)$ under the factorization map is a subset of T^{g-1}. Thus the factorization map induces a mapping from $WM(2, g)$ to T^{g-1}

$$\rho\colon\ WM(2, g) \longrightarrow T^{g-1}$$
$$A(z) \longmapsto (\theta_1, \theta_2, \ldots, \theta_{g-1}).$$

By the uniqueness and exactness of factorization, this mapping ρ is well defined and one-to-one. Because of the degenerate phenomenon of paraunitary matrix

$$V_{\theta_1}(z)V_{\theta_2}(z) = z^{-1}I_2, \tag{11}$$

if $\theta_1 - \theta_2 = k\pi + \pi/2$ for some integer k, this mapping is not onto. The image set will exclude the degenerate points $(\theta_1, \ldots, \theta_{g-1})$ with $\theta_{j_0} - \theta_{j_0+1} = k\pi + \pi/2$ for some $j_0 \in \{1, \ldots, g-2\}$ and $k \in \mathbf{Z}$. Thus one derives

THEOREM 3.3. *The mapping from $WM(2, g)$, the space of real orthogonal wavelet matrices of rank 2 and genus g, to the real torus T^{g-1}*

$$\rho\colon\ WM(2, g) \longrightarrow T^{g-1}$$
$$A(z) \longmapsto (\theta_1, \theta_2, \ldots, \theta_{g-1})$$

is well defined and one-to-one.

Conversely, there is a natural mapping τ from the torus T^{g-1} to $WM(2)$

$$\tau\colon\ T^{g-1} \longrightarrow WM(2)$$
$$(\theta_1, \theta_2, \ldots, \theta_{g-1}) \longmapsto V_{\theta_1}(z)V_{\theta_2}(z)\cdots V_{\theta_{g-1}}(z)\mathbf{H},$$

where $\mathbf{H}$ is the canonical Haar matrix of rank 2, which is exactly

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Again, because of (11), the image set $\tau(T^{g-1}) \neq WM(2, g)$. The degenerate property (11) cancels out two primitive paraunitary matrices at each time, and the image set $\tau(T^{g-1})$ will be

$$\tau\left(T^{g-1}\right) = WM(2, g) \cup WM(2, g-2) \cup WM(2, g-4) \cup \cdots \cup WM(2, \mathrm{rem}),$$

where rem $= 1$ if g is odd or rem $= 2$ if g is even. To make it clear, we define the closure $\overline{WM(2, g)}$ of $WM(2, g)$ by

– if g is odd

$$\overline{WM(2, g)} := \bigcup_{j=1}^{(g+1)/2} WM(2, 2j-1) = WM(2, 1) \cup \cdots \cup WM(2, g)$$

– if g is even

$$\overline{WM(2,g)} := \bigcup_{j=1}^{g/2} WM(2,2j) = WM(2,2) \cup \cdots \cup WM(2,g).$$

Then the mapping τ with the image set restricted to $\overline{WM(2,g)}$ will be onto:

$$\tau\colon\ T^{g-1} \longrightarrow \overline{WM(2,g)}.$$

However, this mapping is not one-to-one because, for example,

$$\begin{aligned}&\left(I_2 - e_1{e_1}^* + e_1{e_1}^* z^{-1}\right)\left(I_2 - e_2{e_2}^* + e_2{e_2}^* z^{-1}\right)\\ &\quad = \left(I_2 - v_1{v_1}^* + v_1{v_1}^* z^{-1}\right)\left(I_2 - v_2{v_2}^* + v_2{v_2}^* z^{-1}\right),\end{aligned}$$

where

$$e_1 = (1,0)', \qquad e_2 = (0,1)'$$

and

$$v_1 = \left(\sqrt{2}/2, -\sqrt{2}/2\right)', \qquad v_2 = \left(\sqrt{2}/2, \sqrt{2}/2\right)'.$$

The structure of the image set of τ suggests that we may divide orthogonal wavelet matrices into two classes, $WM(2,\text{odd})$ (wavelet matrices with odd genus) and $WM(2,\text{even})$ (wavelet matrices with even genus). Inside each class, the following diagram commutes,

$$\begin{array}{ccc} T^{g-1} & \xrightarrow{\tau} & \overline{WM(2,g)} \\ \downarrow i_1 & & \downarrow i_2 \\ T^{g+1} & \xrightarrow{\tau} & \overline{WM(2,g+2)} \end{array},$$

where the inclusion mapping i_1 from T^{g-1} to T^{g+1} is adding a degenerate pair of points (θ_g, θ_{g+1})

$$\begin{aligned} i_1\colon\ T^{g-1} &\longrightarrow T^{g+1}\\ (\theta_1,\ldots,\theta_{g-1}) &\longmapsto (\theta_1,\ldots,\theta_{g-1},\theta_g,\theta_{g+1}),\end{aligned}$$

where $\theta_g - \theta_{g+1} = k\pi + \pi/2$ for some integer k. For example, one can choose $\theta_g = \pi/2$ and $\theta_{g+1} = 0$. And the inclusion mapping i_2 from $\overline{WM(2,g)}$ to $\overline{WM(2,g+2)}$ is a multiplication by z^{-1}

$$\begin{aligned} i_2\colon\ \overline{WM(2,g)} &\longrightarrow \overline{WM(2,g+2)}\\ A(z) &\longmapsto z^{-1}A(z).\end{aligned}$$

So, we have

$$(\tau \circ i_1)(\theta_1,\ldots,\theta_{g-1}) = (i_2 \circ \tau)(\theta_1,\ldots,\theta_{g-1}).$$

From (11), it is easy to verify that the diagram is truly commutative. We formulate it as the following theorem.

THEOREM 3.4. *Define a mapping* τ *from the real torus* T^{g-1} *to* $\overline{WM(2,g)}$, *the closure of* $WM(2,g)$, *by*

$$\tau\colon\ T^{g-1} \longrightarrow \overline{WM(2,g)}$$

$$(\theta_1,\theta_2,\ldots,\theta_{g-1}) \longmapsto V_{\theta_1}(z)V_{\theta_2}(z)\cdots V_{\theta_{g-1}}(z)\mathbf{H},$$

where

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then τ *is well defined and onto, and the following diagram commutes*:

$$\begin{array}{ccc} T^{g-1} & \xrightarrow{\tau} & \overline{WM(2,g)} \\ \downarrow i_1 & & \downarrow i_2 \\ T^{g+1} & \xrightarrow{\tau} & \overline{WM(2,g+2)} \end{array}.$$

For rank $m > 2$, let $WM(m,g)$ be the collection of all real (or complex) orthogonal wavelet matrices of rank m and genus g, and let $WM(m)$ be the collection of all real (or complex) orthogonal wavelet matrices of rank m,

$$WM(m) = \bigcup_{1\le g<\infty} WM(m,g).$$

And let $\mathcal{T}^n$ denote the n-dimensional real (or complex) torus

$$\mathcal{T}^n = \underbrace{S^{m-1}\times S^{m-1}\times\cdots\times S^{m-1}}_{n}.$$

Because of the lack of the uniqueness and exactness of factorization, the factorization map from $WM(m,g)$ to $\mathcal{T}^{g-1}$ is not well defined. However, one can always define the parameter mapping from $\mathcal{T}^n$ to $WM(m)$ by

$$\tau\colon\ \mathcal{T}^n \longrightarrow WM(m)$$

$$(v_1,v_2,\ldots,v_n) \longmapsto V_1(z)V_2(z)\cdots V_n(z)\mathbf{H},$$

where $\mathbf{H}$ is the canonical Haar matrix of rank m. Similar to the case of $m=2$, we can define the closure of $WM(m,g)$ by

$$\overline{WM(m,g)} := WM(m,g)\cup WM(m,g-m)\cup WM(m,g-2m)\cup\cdots\cup WM(m,\text{rem}),$$

where rem is the reminder of g under module m, and $g, g-m, g-2m, \ldots,$ rem is an arithmetical sequence. And one can also generalize Theorem 3.4 to $m > 2$, which will be omitted here.

4. A FAST WAVELET TRANSFORM BASED ON FACTORIZATION

The discrete wavelet transform [24] can be implemented via the multiplication by the polyphase decomposition of the wavelet matrix. In this paper we will refer to the multiplication complexity of the wavelet matrix as the computational complexity of the DWT.

Assume we have an orthogonal wavelet matrix of rank m and genus g, then a direct DWT implementation based on matrix multiplication will have $m^2 g$ multiplications and $(m^2 - m)g$ additions. Thus, in general, for a given orthogonal wavelet matrix, its DWT computational complexity will be $O(m^2 g)$. Note that to make a direct comparison with other DWT implementations, we do not consider the length of the data/signal with which the DWT will be applied. One can equally add the length as an additional parameter and carry out a similar analysis.

In the design problem of wavelet matrices, usually all preimposed conditions, like the approximation order (or, equivalently, the vanishing moments), the orthogonality, and the smoothness, are on the scaling filter only, and the wavelet filters are then constructed from the scaling filter. When $m = 2$, the wavelet filter is uniquely determined (up to a translation) by the scaling filter. When $m > 2$, there is some degree of freedom in the choices of the wavelet filters. It suggests the possibility that from a given scaling filter, one may construct a full wavelet matrix with an asymptotically smaller DWT complexity. Several approaches [15, 20, 30] have been presented to construct the $m - 1$ wavelet filters from the scaling filter. In this paper we will give an optimal solution for the problem of designing wavelet filters, where the optimality is measured in the sense of DWT computational complexity.

First we briefly repeat Heller's procedure to construct a full wavelet matrix from a given scaling filter and a characteristic Haar matrix. A more general construction method is described in [20]. The next theorem is Theorem 4.1 of [15] (see also [27]).

THEOREM 4.1. *Given a scaling filter a_0 of genus g and a characteristic Haar matrix H, there exists an orthogonal wavelet matrix A of genus g whose first row is a_0 and whose characteristic Haar matrix is H.*

The proof in [15] is constructive. It suffices to obtain vectors v_i such that the relationship (8) holds with $d = g - 1$. Without loss of generality, we can assume $k_1 = 0$ (otherwise one can always shift a_0 to get $k_1 = 0$). The factorization (8) now has the form

$$A_0 + A_1 z^{-1} + \cdots + A_{g-1} z^{-g+1} = \left(\prod_{i=1}^{g-1} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1} \right) \right) H,$$

where the first row of each A_k is known. Right-multiplying by H^{-1}, it becomes

$$B_0 + B_1 z^{-1} + \cdots + B_{g-1} z^{-g+1} = \prod_{i=1}^{g-1} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1} \right). \tag{12}$$

Again the first row, of each B_k is known. By comparing the coefficient of z^{-g+1} on both sides of (12), one gets

$$B_{g-1} = v_1 {v_1}^* v_2 {v_2}^* \cdots v_{g-1} {v_{g-1}}^*.$$

The right-hand side is a rank-1 matrix, each of whose rows is proportional to ${v_{g-1}}^*$. Since the first row of B_{g-1} is known, and let it be β, it follows

$$v_{g-1} = \frac{\beta^*}{||\beta||}.$$

Right-multiply (12) by

$$I_m - v_{g-1}v_{g-1}{}^* + v_{g-1}v_{g-1}{}^* z,$$

the inverse of the newly determined prime factor, to obtain

$$C_0 + C_1 z^{-1} + \cdots + C_{g-2} z^{-g+2} = \prod_{i=1}^{g-2} \left(I_m - v_i v_i{}^* + v_i v_i{}^* z^{-1} \right).$$

Again the first row of each C_k is known, and we can repeat the pattern to get $v_{g-2}, v_{g-3}, \ldots, v_1$. It has been shown in [15] that the resulting matrix

$$\left(\prod_{i=1}^{g-1} \left(I_m - v_i v_i{}^* + v_i v_i{}^* z^{-1} \right) \right) H$$

is an orthogonal wavelet matrix of genus g whose first row is a_0 and with characteristic Haar H.

With the above construction procedure, let us examine very carefully the multiplication complexity of each factor in the wavelet matrix factorization. We will prove that, for any scaling filter, one can always construct a full orthogonal wavelet matrix with an $O(mg)$ DWT. We first show that the factorization into primitive paraunitary factors can be further factored into almost diagonal matrices which is similar to the decomposition of Daubechies and Sweldens' [10] for biorthogonal wavelets of rank 2, using the lifting steps.

First, for the primitive paraunitary matrix

$$V_i(z) = I_m - v_i v_i^* + v_i v_i^* z^{-1},$$

a multiplication by $V_i(z)$ has an $O(m)$ complexity because of its special structure as the product of vectors. This $O(m)$ complexity can be also illustrated by the Gaussian elimination method.

Set the unit vector $v_i = (w_1, w_2, \ldots, w_m)^*$; then

$$V_i(z) = \begin{pmatrix} 1 + \bar{w}_1 w_1 (z^{-1} - 1) & \bar{w}_1 w_2 (z^{-1} - 1) & \ldots & \bar{w}_1 w_m (z^{-1} - 1) \\ \bar{w}_2 w_1 (z^{-1} - 1) & 1 + \bar{w}_2 w_2 (z^{-1} - 1) & \ldots & \bar{w}_2 w_m (z^{-1} - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \bar{w}_m w_1 (z^{-1} - 1) & \bar{w}_m w_2 (z^{-1} - 1) & \ldots & 1 + \bar{w}_m w_m (z^{-1} - 1) \end{pmatrix}.$$

For convenience, we define a notation $T_{i,j}(x) = (t_{k,l})_{m \times m}$ with $i \neq j$ to be an $m \times m$ matrix which differs from the identity matrix in only one off-diagonal entry,

$$t_{k,l} := \begin{cases} 1 & \text{if } k = l \\ x & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

Now using the Gaussian elimination method, we get

$$V_i(z) = X(z) \cdot T_{m,m-1}\left(\frac{w_{m-1}}{w_m} \right) \cdot T_{m,m-2}\left(\frac{w_{m-2}}{w_m} \right) \cdots T_{m,1}\left(\frac{w_1}{w_m} \right), \tag{13}$$

where $X(z)$ is of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \bar{w}_1 w_m(z^{-1}-1) \\ 0 & 1 & \dots & 0 & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \bar{w}_{m-1} w_m(z^{-1}-1) \\ -\frac{w_1}{w_m} & -\frac{w_2}{w_m} & \dots & -\frac{w_{m-1}}{w_m} & 1+\bar{w}_m w_m(z^{-1}-1) \end{pmatrix}.$$

Note that if $w_m = 0$, then one can move to w_{m-1} and apply the same procedure.

Continuing the process and using the identity

$$\bar{w}_1 w_1 + \bar{w}_2 w_2 + \cdots + \bar{w}_m w_m = 1$$

we have

$$X(z) = T_{m,1}\left(-\frac{w_1}{w_m}\right) \cdot T_{m,2}\left(-\frac{w_2}{w_m}\right) \cdots T_{m,m-1}\left(-\frac{w_{m-1}}{w_m}\right) \cdot Y(z), \tag{14}$$

where

$$Y(z) = \begin{pmatrix} 1 & 0 & \dots & 0 & \bar{w}_1 w_m(z^{-1}-1) \\ 0 & 1 & \dots & 0 & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \bar{w}_{m-1} w_m(z^{-1}-1) \\ 0 & 0 & \dots & 0 & z^{-1} \end{pmatrix}.$$

Finally

$$Y(z) = T_{1,m}(\bar{w}_1 w_m) \cdot T_{2,m}(\bar{w}_2 w_m) \dots T_{m-1,m}(\bar{w}_{m-1} w_m) \cdot P(z), \tag{15}$$

where

$$P(z) = \begin{pmatrix} 1 & 0 & \dots & 0 & -\bar{w}_1 w_m \\ 0 & 1 & \dots & 0 & -\bar{w}_2 w_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\bar{w}_{m-1} w_m \\ 0 & 0 & \dots & 0 & z^{-1} \end{pmatrix}.$$

Combining (13), (14), and (15), we obtain

LEMMA 4.1. *Let $V_i(z) = I_m - v_i v_i^* + v_i v_i^* z^{-1}$ be a primitive paraunitary matrix with $v_i = (w_1, w_2, \dots, w_m)^*$. Then $V_i(z)$ can be written as a product of elementary almost diagonal and diagonal matrices in the following way,*

$$V_i(z) = T_{m,1}\left(-\frac{w_1}{w_m}\right) \cdots T_{m,m-1}\left(-\frac{w_{m-1}}{w_m}\right) \cdot T_{1,m}(\bar{w}_1 w_m) \dots T_{m-1,m}(\bar{w}_{m-1} w_m)$$
$$\times D \cdot T_{m-1,m}(-\bar{w}_{m-1} w_m) \dots T_{1,m}(-\bar{w}_1 w_m) \cdot T_{m,m-1}\left(\frac{w_{m-1}}{w_m}\right) \dots T_{m,1}\left(\frac{w_1}{w_m}\right),$$

where $D = \text{diag}(1, 1, \dots, 1, z^{-1})$.

Thus the multiplication by $V_i(z)$ can be implemented with $4(m-1)$ multiplications and $4(m-1)$ additions. The next lemma tells us that the DWT complexity of a characteristic Haar matrix is fully determined by its unitary matrix.

LEMMA 4.2. *For a characteristic Haar matrix H, the DWT complexity is equal to the sum of $O(m)$ and the complexity of a multiplication by a matrix U, where U is the unitary matrix in Lemma* 2.2.

Proof. From Lemma 2.2, it suffices to show that the multiplication by the canonical Haar matrix $\mathbf{H}$ has an $O(m)$ complexity. By definition, $\mathbf{H}$ is equal to the product of two matrices

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\frac{1}{m-1}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & \dots & \sqrt{\frac{m}{i^2+i}} & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \sqrt{\frac{m}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & \dots & 1 \\ 1-m & 1 & \dots & \dots & \dots & \dots & 1 \\ \vdots & \ddots & \ddots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & -j & 1 & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & -1 & 1 \end{pmatrix}.$$

The first matrix is diagonal, and a multiplication by it will be $m-1$ multiplications and no additions. For the second matrix, let us denote it by G, and assume

$$G \cdot (x_0, x_1, \dots, x_{m-1})^* = (y_0, y_1, \dots, y_{m-1})^*.$$

Since

$$\begin{aligned} y_0 &= x_0 + x_1 + \cdots + x_{m-1} \\ y_1 &= y_0 - m \cdot x_0 \\ y_2 &= y_1 - (m-1) \cdot (x_1 - x_0) \\ &\vdots \\ y_{m-1} &= y_{m-2} - 2 \cdot (x_{m-2} - x_{m-3}). \end{aligned}$$

So the multiplication of G can be implemented within $m-1$ multiplications and $3m-4$ additions. In total the multiplication by $\mathbf{H}$ has $2m-2$ multiplications and $3m-4$ additions. The lemma is proved. ■

Combining Theorem 4.1 and Lemma 4.2, we obtain

THEOREM 4.2. *Given a scaling filter a_0 of genus g, there exists an orthogonal wavelet matrix A of genus g whose first row is a_0, and its computational complexity of DWT is* $O(mg)$.

Proof. By choosing the canonical Haar matrix $\mathbf{H}$ as the characteristic Haar, we can construct an orthogonal wavelet matrix A of genus g whose first row is a_0 by Theorem 4.1,

$$A(z) = \left(\prod_{i=1}^{g-1} (I_m - v_i v_i^* + v_i v_i^* z^{-1}) \right) \mathbf{H}.$$

Note that each primitive paraunitary matrix

$$I_m - v_i v_i^* + v_i v_i^* z^{-1}$$

can be implemented with $4(m-1)$ multiplications and $4(m-1)$ additions. Since there are exactly $g-1$ prime factors in the factorization, in total they will contribute to $4(m-1)(g-1)$ multiplications and $4(m-1)(g-1)$ additions. Now for the canonical Haar matrix **H**, the unitary matrix $U = I_{m-1}$. Applying Lemma 4.2, it follows that the computational complexity of DWT is $O(mg)$. ■

When designing an orthogonal wavelet matrix, usually the scaling filter is constructed first. In these situations, one can simply choose the canonical Haar matrix (or some other simple structured orthogonal Haar wavelet matrix whose unitary matrix has an $O(m)$ multiplication complexity) as the characteristic Haar matrix and construct the additional $m-1$ wavelet filters (see Theorem 4.1 and 4.2). Then the full wavelet matrix will have an $O(mg)$ DWT. In the case that an orthogonal m-band wavelet matrix is already given, because changing the wavelet filters does not change the approximation order and the smoothness, one can fix the scaling filter and modify the wavelet filters to achieve $O(mg)$ DWT. One can ignore the wavelet filters and construct a new wavelet matrix from the given scaling filter by Theorems 4.1 and 4.2. Or one can employ the eigenfilter approach [35] to find the unit column vectors $v_1, v_2, \ldots, v_d$ in the factorization (8) and force the characteristic Haar matrix to be the canonical Haar matrix (or some other orthogonal Haar wavelet matrix with $O(m)$ DWT). There is some difference between these two approaches. For the eigenfilter approach, the number of prime factors, d, is greater than or equal to $g-1$. When $d > g-1$ (which is the case when the exactness of factorization fails), the multiplication complexity of the prime factors may be larger than $O(mg)$, and the DWT complexity of the constructed wavelet matrix will be larger than the DWT complexity of its characteristic Haar matrix. So with a given full wavelet matrix, the construction method from its scaling filter would seem preferable.

Finally we will discuss a specific example on how to construct a full wavelet matrix from a given scaling filter to have an $O(mg)$ DWT. The fast wavelet transform implementation will also be illustrated.

This example is from [30], a rank 3 scaling filter with approximation order two [2],

$$a_0 = \begin{pmatrix} a_{0,0} \\ a_{0,1} \\ a_{0,2} \\ a_{0,3} \\ a_{0,4} \\ a_{0,5} \end{pmatrix}^* = \begin{pmatrix} 0.58610191307059 \\ 0.91943524640393 \\ 1.25276857973726 \\ 0.41389808692940 \\ 0.08056475359608 \\ -0.25276857973727 \end{pmatrix}^* .$$

[2] A rank m scaling filter is said to have approximation order K if it has a polynomial factor of the form $P^K(z)$, with $P(z) = (1 + z^{-1} + \cdots + z^{-(m-1)})/m$ for maximal possible K.

Note that we use a different normalization (6). In [30], they have

$$\sum_j a_{0,j} = \sqrt{m}.$$

Now a_0 is a scaling filter of genus 2, and the canonical Haar matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}.$$

Using the notation in the proof of Theorem 4.1, the first row of A_1 is

$$(0.41389808692940, \quad 0.08056475359608, \quad -0.25276857973727).$$

Right-multiplying by $\mathbf{H}^{-1}$ (which is equal to $\mathbf{H}^*/m$), we get the first row of B_1, which is

$$\beta = (0.08056475359607, \quad -0.23570226039551, \quad -0.13608276348796).$$

Now normalize this row vector to obtain

$$v_1 = \frac{\beta^*}{||\beta||} = (0.28383930946236, \quad -0.83040739086483, \quad -0.47943593065288)^*.$$

So the full wavelet matrix is equal to

$$\begin{aligned} A(z) &= \left(I_3 - v_1 v_1^* + v_1 v_1^* z^{-1}\right)\mathbf{H} \\ &= \begin{pmatrix} 0.58610191307059 & 0.91943524640393 & 1.25276857973726 \\ -0.20330295558638 & 0.94280904158208 & -0.03239930480915 \\ 0.69911956479291 & -1.08866210790362 & 0.79779083357459 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0.41389808692940 & 0.08056475359608 & -0.25276857973727 \\ -1.21091060678671 & -0.23570226039553 & 0.73950608599570 \\ -0.69911956479291 & -0.13608276348797 & 0.42695403781700 \end{pmatrix} \\ &\quad \times z^{-1}. \end{aligned}$$

For the fast DWT implementation, based on Lemma 4.1,

$$\begin{aligned} & I_3 - v_1 v_1^* + v_1 v_1^* z^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.73205080756882 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.59202761269027 & 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 & -0.13608276348796 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.39812714026031 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.39812714026031 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & -0.13608276348796 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.59202761269027 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.73205080756882 & 1 \end{pmatrix}.$$

And for **H**, it can be factored as

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1}2 & 0 \\ 0 & 0 & \sqrt{3}2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Combining the factorization of the prime factor and the canonical Haar matrix, we derive a fast implementation of the DWT.

5. CONCLUSIONS

In this paper we define a one-to-one mapping ρ from the space of real orthogonal wavelet matrices of rank 2 and genus g to the space

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}.$$

Thus any real orthogonal wavelet matrix can be uniquely identified as a point on a real torus. Meanwhile one can define a mapping τ from the torus to the wavelet space. The image set of τ suggests a partition of all orthogonal wavelet matrices of rank 2. With the same parity on the genus, the mapping τ provides a commutative diagram, as described in Theorem 3.4.

In the general rank m case, one can still define the mapping from the torus (product of S^{m-1}) to the orthogonal wavelet space. It is an onto mapping. However, it is not one-to-one. And there does not exist a well-defined mapping from the orthogonal wavelet space to the torus due to the nonuniqueness and nonexactness of the factorization.

Based on the factorization, we present an $O(mg)$ DWT implementation for a rank m orthogonal wavelet matrix constructed from a given scaling filter. The scaling filter completely determined the approximation order, the orthogonality, and the smoothness of a full wavelet matrix. Thus from the scaling filter, one can always construct a full wavelet matrix with the additional $(m-1)$ wavelet filters whose DWT has an $O(mg)$ computational complexity.

Currently we are investigating the possibility of generalizing the above algebraic structure results to multidimensional wavelet matrices. In an n dimensional wavelet setting, the wavelet matrix will be $n+1$ dimensional. We will report on this in a forthcoming paper.

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