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A Wavelet Approach to Robust Multilevel Solvers for Anisotropic Elliptic Problems

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A wavelet variation of the "Frequency decomposition multigrid method, Part I" (FDMGM) of Hackbusch [*Numer. Math.* **56** (1989) 229–245] is presented. The perfect reconstruction property and the multiresolution structure of wavelets yield the robustness of the additive as well as of the multiplicative version of a two-level method corresponding to any intermediate level in the FDMGM. Aspects of the robustness of the multilevel scheme are discussed. Numerical experiments confirm the theoretical results. The wavelet version of the FDMGM presented here involves wavelet packets which have been used before this primarily in signal processing. © 1994 Academic Press, Inc.

1. INTRODUCTION

An iterative method for solving a linear system originating from a discretization of the *anisotropic* model problem

$$-\varepsilon \frac{\partial^2}{\partial x^2} u(x,y) - \frac{\partial^2}{\partial y^2} u(x,y) + u(x,y) = f(x,y) \quad \text{in } \Omega = (0,s)^2, \qquad (1.1)$$

$$u \text{ periodic on } \Omega. \qquad (1.2)$$

is said to be *robust* if its convergence rate (i.e., the spectral radius of its iteration matrix) is bounded smaller than 1 uniformly in ε and the discretization step size. Without loss of generality we will assume that $0 < \varepsilon \leqslant 1$ ($\varepsilon > 1$ can be modeled by dividing (1.1) by ε).

Figure 1 shows the convergence rate of the wavelet multigrid iteration (see [8]) for solving (1.1), (1.2), plotted as a function of the anisotropy parameter ε. The convergence rate deteriorates if ε approaches zero. One obtains the wavelet multigrid method by a wavelet-Galerkin discretization of the model problem (Section 3) and by using the Mallat transformation (2.21) as canonical restriction, resp., prolongation.

Essentially, there are two ways of obtaining robust multigrid methods. These two ways correspond to the two complementary iterations, the *coarse grid correction* and the *basic iterative method* (BIM), which are the main ingredients of a multigrid process.

One way is to choose a sophisticated BIM as done, for instance, in [22, 26]. However, the performance of the BIM used in these cases (incomplete LU decomposition [26], block versions of the Gauss–Seidel relaxation [22]) is limited; see Chap. 10 in [10], especially for 3D applications.

A modification of the coarse grid correction opens another way toward robustness (see [10, 11, 19]), which can be extended to the 3D case.

Our paper is motivated by and related to the "*Frequency decomposition multigrid method*" of Hackbusch [10]. We will see that the wavelet theory is a natural framework to describe and to analyze this method. Indeed, the perfect reconstruction property of the wavelet system (2.23) enables us to prove the robustness of the two-level methods with respect to any intermediate level in the recursive multilevel process, which is an improvement compared to the original construction of the FDMGM in [10], where only the robustness of the two-level method corresponding to the starting level was verified.

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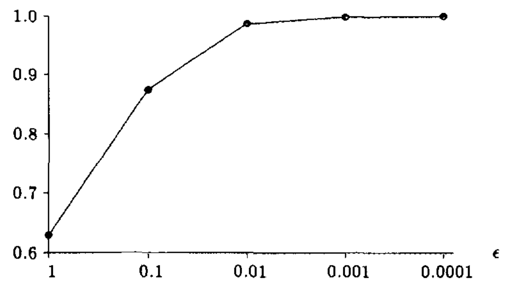

FIG. 1. Convergence rate of the wavelet multigrid iteration as a function of ε.

We do not view this wavelet variation of the FDMGM in the more general setting of additive Schwarz iterations (see [11]), in which the BIM is replaced by subspace corrections (see [27, 28]) which reduce high frequency error components.

There are also other wavelet techniques (wavelet multilevel preconditioner; see [3, 13]) which might be adapted to yield robust solvers.

In the next section we supply the properties of the Daubechies wavelets which we need for our later analysis. We give summaries concerning connection coefficients, Mallat transformations [15], and wavelet packets [20]. Section 3 deals with the wavelet-Galerkin discretization of the model problem (1.1), (1.2). Then the additive coarse grid correction is motivated and formulated in Section 4, and our main result concerning the robustness of the corresponding two-level method is proved. Our theoretical results are confirmed by numerical experiments. As a by-product of our analysis we yield strong Cauchy inequalities for the wavelet packet spaces with respect to H^1-inner products.

The step to a corresponding multilevel method is almost straightforward and done in Section 5. Here we achieve the robustness of the two-level method with respect to any intermediate level. Aspects of the robustness of the (damped) V and of the W-cycle are discussed. Numerical experiments indicate that both cycles are robust. In particular, the experiments indicate the validity of higher order strong Cauchy estimates in this context, which, if valid, would yield the desired robustness result. These desired strong Cauchy inequalities depend, in turn, on delicate (and as yet unknown) properties concerning the Fourier transform of the wavelet filter used in our analysis.

2. WAVELET ANALYSIS

2.1. *Some Properties of Daubechies' Wavelets*

Let φ and ψ be the Daubechies scaling and wavelet function of order N; see [4, 5]. The corresponding scaling and wavelet coefficients are $\{a_k\}_{0\leq k\leq 2N-1}$ and $\{b_k\}_{0\leq k\leq 2N-1}$, respectively. We use the normalization $\sum a_k = 2$. In the Fourier domain, the scaling equation of the scaling function becomes

$$\hat{\varphi}(\xi) = m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right), \tag{2.1}$$

with the Fourier transform $\hat{\varphi}(\xi) := (2\pi)^{-1/2}\int\varphi(x)e^{-ix\xi}dx$ and with

$$m_0(\xi) := \frac{1}{2}\sum_{n=0}^{2N-1} a_n e^{-in\xi}. \tag{2.2}$$

Moreover, m_0 satisfies

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1, \quad \text{for all } \xi \in \mathbf{R}, \tag{2.3}$$

and

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q_N(\xi). \tag{2.4}$$

The trigonometric polynomial Q_N fulfills

$$|Q_N(\xi)| > 0, \quad \xi\in\mathbf{R}, \tag{2.5}$$

a fact we will need later in the paper. Let us set

$$m_1(\xi) := \frac{1}{2}\sum_{n=0}^{2N-1} b_n e^{-in\xi}.$$

Then, we have

$$m_1(\xi) = e^{-i(2N-1)\xi}\left(\frac{1-e^{-i\xi}}{2}\right)^N \overline{Q_N(\xi+\pi)}, \tag{2.6}$$

and

$$\hat{\psi}(\xi) = m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right). \tag{2.7}$$

Furthermore, since $\{\varphi_{0,k}\}_{k\in\mathbf{Z}}$ is an orthonormal set, and $\varphi \in L^1(\mathbf{R})$, one has, (see [17]),

$$\sum_{k\in\mathbf{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 = \frac{1}{2\pi}, \quad \text{for any } \xi\in\mathbf{R}. \tag{2.8}$$

Note that

$$\hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \hat{\psi}(0) = 0. \tag{2.9}$$

2.2. *Connection Coefficients*

From now on, we will assume $N \geq 3$, that is, we restrict ourselves to the differentiable Daubechies scaling and wavelet functions [7]. All the lemmas in this section are supplied for later use.

Let us denote the following *connection coefficients* for the second order derivatives [1, 14],

$$\Gamma_k^h := \int_{\mathbf{R}} \varphi'(x-k)\varphi'(x)\,dx, \tag{2.10}$$

$$\Gamma_k^g := \int_{\mathbf{R}} \psi'(x-k)\psi'(x)\,dx, \tag{2.11}$$

$$\Gamma_k^c := \int_{\mathbf{R}} \varphi'(x-k)\psi'(x)\,dx. \tag{2.12}$$

Since $\operatorname{supp}(\varphi) = \operatorname{supp}(\psi) = [0, 2N-1]$, hence $\Gamma_k^h = \Gamma_k^g = \Gamma_k^c = 0$ for $k \notin [2-2N, 2N-2]$. Moreover, $\Gamma_k^h = \Gamma_{-k}^h$ and $\Gamma_k^g = \Gamma_{-k}^g$ for $k = 2-2N, \ldots, 2N-2$. The following lemma shows that these second order connection coefficients have precise vanishing and nonvanishing moments (see [16] for the similar lemma for first order connection coefficients).

LEMMA 2.1. *Let Γ_k^h, Γ_k^c be defined in* (2.10) *and* (2.12) *for $k \in \mathbf{Z}$. Then,*

(i) $\sum_k k^l \Gamma_k^h = 0$, *for* $l = 0, 1$,
(ii) $\sum_k k^2 \Gamma_k^h \neq 0$,
(iii) $\sum_k k^l \Gamma_k^c = 0$, *for* $l = 0, 1, 2$.

Proof. (i) and (ii) are results from [1] and (iii) can be proved similarly. ■

We now consider the Fourier transforms of these connection coefficients $\Gamma_k^h, \Gamma_k^g, \Gamma_k^c$.

LEMMA 2.2. *For $\xi \in \mathbf{R}$, let us denote*

$$\lambda^h(\xi) := \sum_k \Gamma_k^h e^{-ik\xi}, \tag{2.13}$$

$$\lambda^g(\xi) := \sum_k \Gamma_k^g e^{-ik\xi}, \tag{2.14}$$

$$\lambda^c(\xi) := \sum_k \Gamma_k^c e^{-ik\xi}. \tag{2.15}$$

Then, we have

$$\lambda^h(\xi) = \sum_m |\hat{\varphi}(\xi+2m\pi)|^2 (\xi+2m\pi)^2, \tag{2.16}$$

$$\lambda^g(\xi) = \sum_m |\hat{\psi}(\xi+2m\pi)|^2 (\xi+2m\pi)^2, \tag{2.17}$$

$$\lambda^c(\xi) = \sum_m \hat{\varphi}(\xi+2m\pi)\overline{\hat{\psi}(\xi+2m\pi)} \times (\xi+2m\pi)^2. \tag{2.18}$$

Proof. The statements can be proved by arguing as in [1]. ■

LEMMA 2.3. *Let $\lambda^h(\xi)$ and $\lambda^g(\xi)$ be defined as in the lemma above. Then,*

(i) *$\hat{\varphi}$ has zeros only at $2m\pi$ for $m \in \mathbf{Z} - \{0\}$,*
(ii) *$\lambda^h(\xi) = 0$ iff $\xi = 2m\pi, m \in \mathbf{Z}$,*
(iii) *$\hat{\psi}$ has zeros only at $4m\pi$ for $m \in \mathbf{Z}$,*
(iv) *$\lambda^g(\xi) > 0$ for $\xi \in \mathbf{R}$.*

Remark. Since λ^g is a continuous and 2π-periodic function, so (iv) implies that there is a positive constant c, so that $\lambda^g(\xi) \geq c$ for all $\xi \in \mathbf{R}$.

Proof.

(i) The fact $\hat{\varphi}(2m\pi) = 0$ for $m \in \mathbf{Z} - \{0\}$ follows from (2.8) and (2.9). On the other hand, suppose that there is a $\xi_0 \notin \{2m\pi : m \in \mathbf{Z}\}$, such that $\hat{\varphi}(\xi_0) = 0$. Then by (2.1) and (2.5), we derive $\hat{\varphi}(\xi_0/2) = 0$, as well. Recursively, one can show $\hat{\varphi}(\xi_0/2^k) = 0$, for all $k \in \mathbf{Z}$. Hence by letting k tend to ∞, one has $\hat{\varphi}(0) = 0$ which contradicts to (2.9) because $\hat{\varphi}$ is continuous.
(ii) Suppose that $\lambda^h(\xi) = 0$ for some $\xi \notin \{2m\pi : m \in \mathbf{Z}\}$. Then, by (2.16), one has $\hat{\varphi}(\xi + 2m\pi) = 0$, for all $m \in \mathbf{Z}$. But this contradicts (2.8). Since $\sum_k \Gamma_k^h = 0$, so $\lambda^h(2m\pi) = 0$ for $m \in \mathbf{Z}$.
(iii) This follows from (2.7) and (i).
(iv) Suppose that $\lambda^g(\xi) = 0$ for some $\xi \in [0, 2\pi)$, then by (2.17),

$$|\hat{\psi}(\xi+2m\pi)|^2 (\xi+2m\pi)^2 = 0, \quad \text{for all } m \in \mathbf{Z}. \tag{2.19}$$

In particular, $m = 1$ implies $\hat{\psi}(\xi+2\pi)(\xi+2\pi) = 0$. But this is impossible by (iii). Therefore, (iv) is proved, since λ^g is 2π-periodic. ■

LEMMA 2.4. *Let λ^h and λ^c be defined by* (2.13) *and* (2.15), *respectively. Then,*

$$\lim_{\xi \to 0} \frac{\lambda^c(\xi)}{\lambda^h(\xi)} = 0. \tag{2.20}$$

Proof. By the definitions of λ^h and λ^c, the lemma follows from Lemma 2.1 and the l'Hospital rule. ■

Remark. Since λ^h and λ^c are 2π – periodic functions, it follows that (2.20) holds at all points $2k\pi, k \in \mathbf{Z}$.

2.3. Mallat Transformations

The (periodic) *Mallat transformations* $h, g : \mathbf{R}^{2n} \to \mathbf{R}^n$ of a vector $v \in \mathbf{R}^{2n}$ are defined by

$$(hv)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} a_l v_{l+2k}, \quad k = 0, \ldots, n-1, \tag{2.21}$$

$$(gv)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} b_l v_{l+2k}, \quad k = 0, \ldots, n-1, \tag{2.22}$$

where we extend v periodically, i.e., $v_l = v_{l+2n}$.

The Mallat transformations satisfy (see [4, 15]),

$$h^t h + g^t g = I, \tag{2.23}$$

$$hh^t = gg^t = I,$$

$$gh^t = hg^t = 0. \tag{2.24}$$

We use I to denote the identity matrix of appropriate size throughout this paper.

2.4. Wavelet Packets

Wavelet packets were originally introduced in [20] to improve the frequency resolution of signals achieved by a wavelet analysis. We will need wavelet packets for space decomposition later in our multigrid scheme.

The *wavelet packets* $\{\psi^\kappa : \kappa \in \mathbf{N}_0, \kappa \text{ is binary}\}$ (where $\mathbf{N}_0$ denotes the nonnegative integers) are recursively defined by

$$\psi^{\kappa 0}(x) := \sum_{m=0}^{2N-1} a_m \psi^\kappa (2x - m), \quad \kappa = 1, 10, 11, \ldots, \tag{2.25}$$

and

$$\psi^{\kappa 1}(x) := \sum_{m=0}^{2N-1} b_m \psi^\kappa (2x - m), \quad \kappa = 1, 10, 11, \ldots, \tag{2.26}$$

where we set $\psi^0 = \varphi$ and $\psi^1 = \psi$. We note the natural definition

$$W_j^\kappa := \text{closure} \left(\text{span}\left\{\psi_{j,m}^\kappa (x) := 2^{j/2}\psi^\kappa (2^j x - m) : m \in \mathbf{Z}\right\}\right).$$

The scaling function space $V_j := W_j^0$ has now the following orthogonal decomposition into wavelet packet spaces; see [5, 18]:

$$V_j = \bigoplus_{\kappa=0}^{2^l - 1} W_{j-l}^\kappa, \quad \kappa \text{ is binary and } l \in \mathbf{N}.$$

3. WAVELET-GALERKIN DISCRETIZATION OF THE MODEL PROBLEM

3.1. Variational Formulation

We introduce the Sobolev space $H_p^1(\Omega), \Omega = (0,s)^2$, with periodic boundary conditions,

$$H_p^1 = H_p^1(\Omega) := \{v \in L^2(\Omega) : v_x, v_y \in L^2(\Omega), v(0,y) = v(s,y), v(x,0) = v(x,s)\}.$$

The weak or variational formulation of the model problem (1.1), (1.2), reads:

$$\text{find } u \in H_p^1 : \mathscr{A}(u,v) = \int_\Omega f v \, dx dy, \quad \text{for all } v \in H_p^1, \tag{3.1}$$

where $\mathscr{A}$ is the H_p^1-elliptic bilinear form

$$\mathscr{A}(u,v) = \int_\Omega \left(\varepsilon u_x v_x + u_y v_y + uv\right) dx dy. \tag{3.2}$$

Due to the Lax–Milgram theorem [2] (3.1) has a unique solution u. For a wavelet-Galerkin discretization of (3.1), we assume that s in (1.1) is an integer greater than $4N - 3$. Further, we introduce the wavelet-Galerkin spaces

$$V_l^p = V_l^p(0,s) := \left\{v \in L^2(0,s) : v(x) = \sum_{k\in\mathbf{Z}} c_k \varphi_{l,k}(x), \; x \in [0,s], \text{ and } c_k = c_{k+2^l s}\right\}, \tag{3.3}$$

and for later use

$$W_l^p = W_l^p(0,s) := \left\{v \in L^2(0,s) : v(x) = \sum_{k\in\mathbf{Z}} c_k \psi_{l,k}(x), \; x \in [0,s], \text{ and } c_k = c_{k+2^l s}\right\}. \tag{3.4}$$

Obviously, V_l^p and W_l^p have the dimension $n_l = 2^l s$. The wavelet-Galerkin approximation $u_l \in V_l^p \otimes V_l^p$ to u is the unique solution of

$$\mathscr{A}(u_l, v_l) = \int_\Omega f v_l \, dx dy, \quad \text{for all } v_l \in V_l^p \otimes V_l^p.$$

A convergence proof of $u_l \to u$ is given in [24]. We expand u_l,

$$u_l(x,y) = \sum_{i,j\in\mathbf{Z}} u_{i,j}^l \varphi_{l,i}(x) \varphi_{l,j}(y),$$

where the expansion coefficients are periodic in each index with period n_l, and we define $f_{i,j}^l = \int_\Omega f(x,y)\varphi_{l,i}(x) \times \varphi_{l,j}(y) dx dy$. If we now order the coefficients $u_{i,j}^l$ and $f_{i,j}^l, 0 \leq i,j \leq n_l - 1$, lexicographically and denote the resulting vectors U_l and F_l, respectively, then we have the following linear system for the n_l^2 unknowns U_l,

$$A_l U_l = F_l, \tag{3.5}$$

where ($\otimes$ denotes the tensor product of spaces, operators and vectors)

$$A_l = \varepsilon c_l^h \otimes I + I \otimes c_l^h + I \otimes I \tag{3.6}$$

is the Galerkin approximation (stiffness matrix) of $\mathscr{A}$ in $V_l^p \otimes V_l^p$. In (3.6), c_l^h is a symmetric and circulant $n_l \times n_l$ matrix [6], which is completely determined by the connection coefficients Γ_k^h (2.10),

$$c_l^h = \delta_l^{-2} \begin{pmatrix} \Gamma_0^h & \Gamma_1^h & \cdots & \Gamma_p^h & 0 & 0 & \cdots & 0 & \Gamma_p^h & \cdots & \Gamma_2^h & \Gamma_1^h \\ \Gamma_1^h & \Gamma_0^h & \cdots & \cdots & \Gamma_p^h & 0 & \cdots & 0 & 0 & \Gamma_p^h & \cdots & \Gamma_2^h \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Gamma_1^h & \cdots & \Gamma_p^h & 0 & 0 & 0 & \cdots & \Gamma_p^h & \cdots & \cdots & \Gamma_1^h & \Gamma_0^h \end{pmatrix}, \tag{3.7}$$

where $p = 2N - 2$ and $\delta_l := 2^{-l}$ is the *discretization step size*.

3.2. Some Properties of Circulant Matrices

Let $(\Gamma_0\Gamma_1 \cdots \Gamma_p 0 \cdots 0 \Gamma_{-p} \cdots \Gamma_{-1})$ be the first row of the $n \times n$ circulant matrix $B, (n \geqslant 2p + 1)$. We note that

$$B = \mathrm{Cir}_n \left(\Gamma_0\Gamma_1 \cdots \Gamma_p 0 \cdots 0 \Gamma_{-p} \cdots \Gamma_{-1}\right),$$

where $\mathrm{Cir}_n(c_1, \ldots, c_n)$ is the $n \times n$ circulant matrix generated by the first row $(c_1, \ldots, c_n)$. Then from [6], we know that the eigenvalues of B are

$$\lambda_\alpha = \sum_{k=-p}^{p} \Gamma_k e^{-i2\pi\alpha k/n}, \quad \alpha = 0, \ldots, n-1, \tag{3.8}$$

and their corresponding l^2-orthonormal eigenvectors $v_\alpha, \alpha = 0, \ldots, n-1$, are

$$(v_\alpha)_k = \frac{1}{\sqrt{n}} e^{-i2\pi\alpha k/n}, \quad k = 0, \ldots, n-1. \tag{3.9}$$

Remark. All circulant matrices of the same size have the same eigenvectors. In particular, the positive definite matrix A_l in (3.6) has the eigenvalues

$$\Lambda_{\alpha,\beta} = \delta_l^{-2} \left(\varepsilon\lambda_\alpha + \lambda_\beta\right) + 1 > 0 \tag{3.10}$$

with corresponding eigenvectors $v_\alpha \otimes v_\beta$ for $\alpha, \beta = 0, \ldots, n-1$.

4. THE TWO-LEVEL METHOD

In this section, we propose a two-level method to solve the linear system (3.5), and prove its robustness.

4.1. Motivation of the Method and Main Result

Assume that one wants to solve (3.5) at level l. Following the philosophy of multigrid methods [9, 25], we first apply a classical relaxation process (Richardson, Jacobi, or Gauss–Seidel) as BIM ν times with initial guess U_l^0. We obtain an approximate solution U_l^ν. In general, the BIM only reduces error components well in the direction of eigenvectors corresponding to large eigenvalues. In the presence of strong anisotropy, that is, ε small compared to 1, this means the error $E_l^\nu = U_l - U_l^\nu$ may contain not only a part which is low frequency in both the x and y-directions, but also a part which is high frequency in x-direction. This can be seen by (3.9) and (3.10). For a more detailed discussion we refer to [10]. Consequently, we approximate the error E_l^ν in both spaces $V_{l-1}^p \otimes V_{l-1}^p$ and $W_{l-1}^p \otimes V_{l-1}^p$ instead of only in $V_{l-1}^p \otimes V_{l-1}^p$, as in a standard multigrid procedure. In the abstract setting of [27, 28], the resulting coarse grid correction

$$U_l^\nu \leftarrow U_l^\nu + \left[(h' \otimes h') A_{l-1}^{-1} (h \otimes h) + (g' \otimes h') \left(A_{l-1}^g\right)^{-1} (g \otimes h)\right] (F_l - A_l U_l^\nu) \tag{4.1}$$

may be viewed as an *additive subspace correction* with respect to the orthogonal subspaces $V_{l-1}^p \otimes V_{l-1}^p$ and $W_{l-1}^p \otimes V_{l-1}^p$.

The $n_{l-1}^2 \times n_{l-1}^2$ dimensional matrix A_{l-1}^g is a Galerkin approximation (stiffness matrix) of the bilinear form $\mathscr{A}$ (3.2) with respect to $W_{l-1}^p \otimes V_{l-1}^p$,

$$\begin{aligned} A_{l-1}^g &:= (g \otimes h) A_l (g' \otimes h') \\ &= \varepsilon c_{l-1}^g \otimes I + I \otimes c_{l-1}^h + I \otimes I, \end{aligned} \tag{4.2}$$

where the circulant matrix $c_{l-1}^g = g c_l^h g'$ of dimension $n_{l-1} \times n_{l-1}$ is built by the $2N - 1$ connection coefficients $\Gamma_k^g, k = 0, \ldots, 2N-2$, of the wavelet ψ (2.11),

$$c_{l-1}^g = \delta_{l-1}^{-2} \mathrm{Cir}_{n_{l-1}} \left(\Gamma_0^g \cdots \Gamma_{2N-2}^g 0 \cdots 0 \Gamma_{2N-2}^g \cdots \Gamma_1^g\right).$$

For our further considerations, we need the Euclidean (spectral) norm $\|\cdot\|$ as well as the energy-norm relative to (3.5), $\|x\|_{A_l} = \|A_l^{1/2} x\|$, with the associated matrix norm $\|B\|_{A_l} = \|A_l^{1/2} B A_l^{-1/2}\|$.

In order to show that the robustness of the proposed two-level method does not depend on a suitably chosen BIM we make only general assumptions on the iteration matrix $S_l = I - L_l^{-1} A_l$ of the BIM,

$$\left\|A_l^{1/2} S_l^\nu\right\|_{A_l} \leqslant C_s \eta(\nu) \delta_l^{-1}, \quad \eta(\nu) \to 0, \text{as } \nu \to \infty, \tag{4.3}$$

where the constant C_s and the function η depend neither on δ_l nor on ε. Further,

$$\|S_l^\nu\|_{A_l} \leqslant 1. \tag{4.4}$$

The assumption (4.3) is usually called the *smoothing property* of the BIM; see [9].

Remark. With a suitably chosen relaxation parameter $\beta \in \mathbf{R}$, the pointwise versions of the Richardson ($L_l = \beta^{-1}I$) and the damped Jacobi iteration ($L_l = \beta^{-1}\text{diag}(A_l)$) satisfy (4.3) as well as (4.4) with $\eta(\nu) = \sqrt{\eta_0(2\nu)}, \eta_0(\nu) = \nu^\nu/(\nu+1)^{\nu+1}$; see, e.g., [9, Chap. 6.2].

One step of the robust wavelet-based method (or frequency decomposition method in the notation of [10]) is as follows: after ν steps of the BIM, perform one coarse grid correction step (4.1). If we denote the iteration matrix of this method by $M_l(\nu)$, then

$$M_l(\nu) = \Big[I - (h^t \otimes h^t) A_{l-1}^{-1} (h \otimes h) A_l - (g^t \otimes h^t) \left(A_{l-1}^g\right)^{-1} (g \otimes h) A_l\Big] S_l^\nu. \quad (4.5)$$

Now, we are ready to state the robustness of the method (4.5) (for convenience, we refer to an iterative method by its iteration matrix). The proof of the following theorem will cover the whole next section.

THEOREM 4.1. *Let $M_l(\nu)$ be the iteration matrix defined in* (4.5). *Assume that the BIM S_l satisfies* (4.3) *and* (4.4). *Then, there exist two positive constants C and σ which are both independent of ε and the discretization step size δ_l, such that*

$$\|M_l(\nu)\|_{A_l} \leqslant C\eta(\nu) + \sqrt{r(\varepsilon, \delta_l)}, \quad (4.6)$$

where the function $r(\cdot,\cdot)$ is positive and uniformly bounded by $\sigma < 1$. Moreover, $r(\varepsilon,\delta_l) \to 0$ for any fixed $\delta_l > 0$ as ε tends to zero.

The constant σ will be given more explicitly in Lemma 4.8 in the next section.

COROLLARY 4.2. *The wavelet two-level method* (4.5) *is robust, that is, for any $\tilde\rho \in (\sqrt{\sigma}, 1)$, there exists a positive integer $\nu_{\tilde\rho}$ such that the spectral radius $\rho(M_l(\nu))$ of $M_l(\nu)$ satisfies*

$$\rho(M_l(\nu)) \leqslant \tilde\rho \quad \textit{for all } \nu \geqslant \nu_{\tilde\rho}.$$

Figure 2 shows numerical approximations to the spectral radius of the two-level iteration matrix $M_l(2)$ (4.5) for the Daubechies wavelet with order $N = 3$. The damped Jacobi iteration was used as BIM with damping factor $\beta = \Gamma_0^h/\sum_k |\Gamma_k^h|$. For δ_l fixed, the spectral radius decreases in ε because the function r in (4.6) does.

4.2. Proof of the Robustness of the Two-Level Method

For the proof of Theorem 4.1, we supply several preparatory results. We will first prove the following kind of approximation property for the additive coarse grid correction. For convenience, we write n instead of n_l and suppose $n \geqslant 2s$.

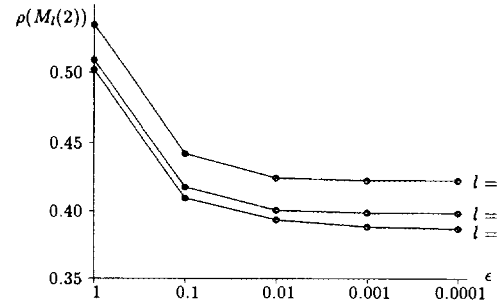

FIG. 2. Convergence rate of the two-level method $M_l(2)$ as a function of the anisotropy parameter ε for several levels l with corresponding dimensions $n_l = 2^l s, s = 16$. The underlying Daubechies wavelet has order $N = 3$.

LEMMA 4.3. *Let A_l be the discretization matrix defined in* (3.6). *We define a linear mapping $U : \mathbf{R}^{n^2} \to \mathbf{R}^{n^2/2}$, by*

$$U := \begin{pmatrix} h \otimes h \\ g \otimes h \end{pmatrix}. \quad (4.7)$$

Then, one has

$$\|(I - U^t U) A_l^{-1}\| \leqslant C\delta_l^2, \quad (4.8)$$

for some constant C independent of ε and δ_l.

Proof. 1. Note that $I - U^t U = Q^t Q$ with

$$Q := \begin{pmatrix} h \otimes g \\ g \otimes g \end{pmatrix} \quad (4.9)$$

by (2.23). Now, given any vector $f \in \mathbf{R}^{n^2}$, one has

$$\|Q^t Q A_l^{-1} f\|^2 = \sum_{\alpha,\beta=0}^{n-1} \sum_{\mu,\nu=0}^{n-1} \Lambda_{\alpha,\beta}^{-1} \Lambda_{\mu,\nu}^{-1} \langle f, v_\alpha \otimes v_\beta \rangle \times \overline{\langle f, v_\mu \otimes v_\nu \rangle} \cdot \left\langle Q(v_\alpha \otimes v_\beta), Q\left(\overline{v_\mu \otimes v_\nu}\right)\right\rangle, \quad (4.10)$$

where the $\Lambda_{\alpha,\beta}$'s are the eigenvalues of A_l in (3.10), and $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product in the real space. Clearly,

$$\left\langle Q(v_\alpha \otimes v_\beta), Q\left(\overline{v_\mu \otimes v_\mu}\right)\right\rangle = \left\langle hv_\alpha, h\bar v_\mu \right\rangle \langle gv_\beta, g\bar v_\nu \rangle + \left\langle gv_\alpha, g\bar v_\mu \right\rangle \langle gv_\beta, g\bar v_\nu \rangle.$$

2. We now analyse the terms $\langle hv_\alpha, h\bar v_\mu \rangle$ and $\langle gv_\beta, g\bar v_\nu \rangle$. By the definition of h (2.21) and (3.9), one has

$$(hv_\alpha)_k = \sqrt{\frac{2}{n}} e^{-i2\pi 2\alpha k/n} m_0 \left(\frac{2\alpha\pi}{n}\right),$$

for $k = 0, \dots, n/2 - 1$. Similarly, one obtains for $k = 0, \dots, n/2 - 1$, that

$$(gv_\alpha)_k = \sqrt{\frac{2}{n}} e^{-i2\pi 2\alpha k/n} m_1 \left(\frac{2\alpha\pi}{n}\right).$$

Therefore,

$$\left\langle hv_\alpha, h\bar{v}_\mu \right\rangle = \frac{2}{n} \sum_{k=0}^{n/2-1} e^{-i2\pi 2(\alpha-\mu)k/n} m_0 \left(\frac{2\alpha\pi}{n}\right) \overline{m_0 \left(\frac{2\mu\pi}{n}\right)}, \quad (4.11)$$

and

$$\langle gv_\beta, g\bar{v}_\nu \rangle = \frac{2}{n} \sum_{k=0}^{n/2-1} e^{-i2\pi 2(\beta-\nu)k/n} m_1 \left(\frac{2\beta\pi}{n}\right) \overline{m_1 \left(\frac{2\nu\pi}{n}\right)}. \quad (4.12)$$

If we use $S_n^{\alpha,\mu}$ to denote the sum over k in (4.11) then an easy calculation gives

$$S_n^{\alpha,\mu} = \begin{cases} 1, & \alpha - \mu = kn/2, k \in \mathbf{Z}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.13)$$

The products (4.11) and (4.12) can now be expressed by

$$\left\langle hv_\alpha, h\bar{v}_\mu \right\rangle = S_n^{\alpha,\mu} m_0 \left(\frac{2\pi\alpha}{n}\right) \overline{m_0 \left(\frac{2\pi\mu}{n}\right)},$$
$$\langle gv_\beta, g\bar{v}_\nu \rangle = S_n^{\beta,\nu} m_1 \left(\frac{2\pi\beta}{n}\right) \overline{m_1 \left(\frac{2\pi\nu}{n}\right)}, \quad (4.14)$$

which enable us to estimate the sum (4.10) by

$$\|Q^t Q A_l^{-1} f\|^2 \leq C \|f\|^2 \cdot \max \frac{|\langle gv_\beta, g\bar{v}_\nu \rangle|}{\Lambda_{\alpha,\beta}\Lambda_{\mu,\nu}}. \quad (4.15)$$

3. By (3.10), (3.8), and (2.13), we have

$$\Lambda_{\alpha,\beta} = \delta_l^{-2} \left(\varepsilon\lambda^h \left(2\pi\frac{\alpha}{n}\right) + \lambda^h \left(2\pi\frac{\beta}{n}\right)\right) + 1. \quad (4.16)$$

In step 4, we will prove the following.

LEMMA 4.4. *We have that*

$$\sup_\xi \frac{|m_1(\xi)|}{\lambda^h(\xi)} = C < \infty. \quad (4.17)$$

Using (4.17), (4.16), and (4.14), one derives that

$$\frac{|\langle gv_\beta, g\bar{v}_\nu \rangle|}{\Lambda_{\alpha,\beta}\Lambda_{\mu,\nu}} \leq C\delta_l^4. \quad (4.18)$$

Lemma 4.3 follows from (4.15) and (4.18).

4. Finally, we prove Lemma 4.4. By part (ii) of Lemma 2.3, $\lambda^h(\xi)$ has zeros only at $2m\pi, m \in \mathbf{Z}$. Moreover, by Lemma 2.1, λ^h has second order zeros at these points. But when $N \geq 3$, which we are assuming, $m_1(\xi)$ has zeros of at least order 3 at $2m\pi, m \in \mathbf{Z}$. Hence the lemma is proved since both λ^h and m_1 are continuous 2π-periodic functions. ■

COROLLARY 4.5. *Let U be defined as in* (4.7). *Then,*

$$\left\|A_l^{1/2}(I - U^t U)A_l^{-1}\right\| \leq C\delta_l, \quad (4.19)$$

for some constant C which is independent of ε and δ_l.

Proof. Since $\|A_l^{1/2}\| \leq C\delta_l^{-1}$, (4.19) follows from (4.8). ■

LEMMA 4.6. *Let S_l be the iteration matrix of a BIM for solving* (3.5) *satisfying* (4.3). *Then,*

$$\|(I - U^t U)S_l^\nu\|_{A_l} \leq C\eta(\nu), \quad (4.20)$$

where C is a constant independent of ε, δ_l.

Proof. First,

$$\|(I - U^t U)S_l^\nu\|_{A_l} = \left\|A_l^{1/2}(I - U^t U)S_l^\nu A_l^{-1/2}\right\| \leq C\delta_l \left\|A_l S_l^\nu A_l^{-1/2}\right\|,$$

by (4.19). Therefore, the lemma follows readily from (4.3). ■

Remark. The estimate (4.20) states analytically that a BIM satisfying (4.3) reduces those error components well corresponding to the subspace $\{z \in \mathbf{R}^{n^2} : U^t U z = z\}$. In the less precise words of the beginning of Section 4.1: the BIM damps strongly error components which are smooth in the directions of both coordinates axes as well as which are high frequency in the x and smooth in the y-directions.

We define the following auxiliary matrix:

$$\tilde{M}_l(\nu) := \left(I - U^t (U A_l U^t)^{-1} U A_l\right) S_l^\nu. \quad (4.21)$$

The matrix $\tilde{M}_l(\nu)$ in fact is the iteration matrix of a two-level method between the spaces $V_l^p \otimes V_l^p$ and $V_l^p \otimes V_{l-1}^p$. This two-level method corresponds to the semicoarsening of grids in [19]. Further, $M_l(\nu)$ can be viewed as an approximation to $\tilde{M}_l(\nu)$: one step of an (undamped) block Jacobi method to approximate the inverse of UA_lU^t in (4.21) yields $M_l(\nu)$. Moreover, if ε is equal to zero, then $M_l(\nu) = \tilde{M}_l(\nu)$. This fact explains the montonic decrease of $\rho(M_l(2))$ in ε (see Fig. 2).

THEOREM 4.7. *Let $\tilde{M}_l(\nu)$ be defined as in* (4.21). *Under the assumptions of Lemma* 4.6, *we have*

$$\|\tilde{M}_l(\nu)\|_{A_l} \leq C\eta(\nu),$$

where C is a constant independent of ε, δ_l.

Proof. Note that $\tilde{M}_l(\nu) = (I - U^t(UA_lU^t)^{-1}UA_l)(I - U^tU)S_l^\nu$. Hence,

$$\|\tilde{M}_l(\nu)\|_{A_l} \leq \left\|I - U^t(UA_lU^t)^{-1}UA_l\right\|_{A_l} \|(I - U^tU)S_l^\nu\|_{A_l}.$$

The statement follows from (4.20) and from $\|I - U^t \times (UA_lU^t)^{-1}UA_l\|_{A_l} = 1$ which holds true because $I - A_l^{1/2}U^t(UA_lU^t)^{-1}UA_l^{1/2}$ is an orthogonal projector with respect to the Euclidean inner product. ■

To finish the proof of Theorem 4.1, we only need to estimate the difference of $M_l(\nu)$ and $\tilde{M}_l(\nu)$. For this, we introduce the following block diagonal matrix:

$$D_l = \begin{pmatrix} A_{l-1} & 0 \\ 0 & A_{l-1}^g \end{pmatrix}.$$

Here A_{l-1} and A_{l-1}^g are defined in (3.6) and (4.2), respectively. Therefore,

$$\tilde{M}_l(\nu) - M_l(\nu) = U^t\left((UA_lU^t)^{-1} - D_l^{-1}\right)UA_lS_l^\nu,$$

which is estimated by

$$\|\tilde{M}_l(\nu) - M_l(\nu)\|_{A_l} \leq \left\|U^t\left((UA_lU^t)^{-1} - D_l^{-1}\right)UA_l\right\|_{A_l} \|S_l^\nu\|_{A_l}. \quad (4.22)$$

By (4.4), $\|S_l^\nu\|_{A_l} \leq 1$, and

$$\begin{aligned}\left\|U^t\left((UA_lU^t)^{-1} - D_l^{-1}\right)UA_l\right\|_{A_l} &= \left\|A_l^{1/2}U^t\left((UA_lU^t)^{-1} - D_l^{-1}\right)UA_l^{1/2}\right\| \\ &= \rho\left(I - D_l^{-1}UA_lU^t\right) \\ &= \rho\begin{pmatrix} 0 & A_{l-1}^{-1}C_{l-1} \\ \left(A_{l-1}^g\right)^{-1}C_{l-1}^t & 0 \end{pmatrix} \\ &= \rho\left(A_{l-1}^{-1}C_{l-1}\left(A_{l-1}^g\right)^{-1}C_{l-1}^t\right)^{1/2}, \end{aligned} \quad (4.23)$$

where $C_{l-1} = (h\otimes h)A_l(g^t\otimes h^t) = \varepsilon(hc_l^hg^t)\otimes I$, by (2.24). The last equality in (4.23) follows from, for instance, Lemma 5.2.3 in [12]. One observes that

$$\begin{aligned}c_{l-1} &:= hc_l^hg^t \\ &= \delta_{l-1}^{-2}\mathrm{Cir}_{n/2}(\Gamma_0^c\cdots\Gamma_{2N-2}^c0\cdots0\Gamma_{2-2N}^c\cdots\Gamma_{-1}^c).\end{aligned} \quad (4.24)$$

(See (2.12) for the connection coefficients Γ_k^c.) Let us denote the eigenvalues of c_{l-1} by λ_α^c, for $\alpha = 0, \ldots, n/2 - 1$. Then,

$$\rho\left(A_{l-1}^{-1}C_{l-1}\left(A_{l-1}^g\right)^{-1}C_{l-1}^t\right) = \max_{\alpha,\beta}\frac{\varepsilon^2|\lambda_\alpha^c|^2}{\Lambda_{\alpha,\beta}^h\Lambda_{\alpha,\beta}^g} \leq \max_\alpha\frac{|\lambda_\alpha^c|^2}{\lambda_\alpha^h\lambda_\alpha^g} \leq \sigma < 1, \quad (4.25)$$

where the last two inequalities are valid because of Lemma 4.8 (below). Combining (4.23) and (4.25), one can derive from (4.22) that

$$\|\tilde{M}_l(\nu) - M_l(\nu)\|_{A_l} \leq \sqrt{\sigma} < 1. \quad (4.26)$$

Thus, Theorem 4.1 follows from Theorem 4.7 and (4.26).

Remark. If we combine the BIM S_l with the *multiplicative* version of the subspace correction (4.1), then we yield a two-level method with the following iteration matrix

$$\begin{aligned}\mathcal{M}_l(\nu) &:= \left[I - (g^t\otimes h^t)\left(A_{l-1}^g\right)^{-1}(g\otimes h)A_l\right] \\ &\quad\times\left[I - (h^t\otimes h^t)A_{l-1}^{-1}(h\otimes h)A_l\right]S_l^\nu \\ &= \left[I - U^tT_l^{-1}UA_l\right]S_l^\nu,\end{aligned} \quad (4.27)$$

where

$$T_l^{-1} = \begin{pmatrix} A_{l-1}^{-1} & 0 \\ -\left(A_{l-1}^g\right)^{-1}(g\otimes h)A_l(h^t\otimes h^t)A_{l-1}^{-1} & \left(A_{l-1}^g\right)^{-1} \end{pmatrix}.$$

Performing a calculation like (4.23) with T_l^{-1} instead of D_l^{-1} shows that Theorem 4.1 holds also for $\mathcal{M}_l(\nu)$, however, with a sharper estimate: The inequality (4.6) has to be replaced by

$$\|\mathcal{M}_l(\nu)\|_{A_l} \leq C\eta(\nu) + r(\varepsilon, \delta_l), \quad (4.28)$$

and Corollary 4.2 has to be modified accordingly.

LEMMA 4.8. *Let $\lambda^h(\xi)$, $\lambda^g(\xi)$, and $\lambda^c(\xi)$ be defined as in* (2.13), (2.14), *and* (2.15). *Then*

$$\sigma := \sup_{0\leq\xi<2\pi}\frac{|\lambda^c(\xi)|^2}{\lambda^h(\xi)\lambda^g(\xi)} < 1. \quad (4.29)$$

Proof. The above lemma is a special case of Lemma 5.2. ■

For the case $N = 3$, a numerical calculation shows that the constant σ in Lemma 4.8 satisfies $\sigma < 0.3$.

As a consequence of Lemma 4.8 we obtain the following strong Cauchy inequality.

THEOREM 4.9. *Let V_l^p and W_l^p be the spaces defined in* (3.3) *and* (3.4), *respectively. Then for any given functions $v \in V_l^p$ and $w \in W_l^p$, one has*

$$\left|\int_0^s v'(x)w'(x)\,dx\right|^2 \leqslant \sigma\left(\int_0^s |v'(x)|^2 dx\right)\left(\int_0^s |w'(x)|^2 dx\right), \quad (4.30)$$

where $\sigma < 1$ is the constant in Lemma 4.8.

5. MULTILEVEL SCHEME

5.1. Definition and Notation

The multilevel scheme corresponding to the two-level method M_l (4.5) can be derived in an almost straightforward manner. To define the complete multilevel procedure for solving (3.5) we denote (κ is a binary number)

$$A_{l,\kappa} := \varepsilon c_l^\kappa \otimes I + I \otimes c_l^h + I \otimes I, \quad (5.1)$$

where

$$c_l^\kappa = \delta_l^{-1}\mathrm{Cir}_{n_l}\left(\Gamma_0^\kappa \cdots \Gamma_{2N-2}^\kappa 0 \cdots 0 \Gamma_{2N-2}^\kappa \cdots \Gamma_1^\kappa\right), \quad (5.2)$$

with

$$\Gamma_\mu^\kappa = \int_{\mathbf{R}} (\psi^\kappa)'(x-\mu)\,(\psi^\kappa)'(x)\,dx, \quad \mu = 0, \ldots, 2N-2,$$

(see (2.10) and (2.11)). Due to the recursive definition of the wavelet packets, their connection coefficients can be easily computed from the connection coefficients of the scaling function φ.

With the definition (5.1) we have $A_l = A_{l,0}$ and $A_l^g = A_{l,1}$. The matrices $A_{l,\kappa}$ are Galerkin approximations (stiffness matrices) of the bilinear form $\mathscr{A}$ (3.2) relative to the spaces $W_l^\kappa \otimes V_l^p$ (all wavelet packet spaces in this section are understood in their periodic modifications analogously to (3.3) and (3.4), for convenience we will however subpress the superscript p). Therefore, they are all positive definite.

We denote the iteration matrix of the BIM with respect to $A_{l,\kappa}$ by $S_{l,\kappa} = I - L_{l,\kappa}^{-1}A_{l,\kappa}$ and we suppose that all needed connection coefficients are precomputed and stored. Our procedure has four variable quantities: l is the varying level, $0 \leqslant l \leqslant L$, where L is the level corresponding to the smallest discretization step size; κ is a binary integer at level l; w is an approximate solution; b is a right-hand side or a defect. We then define the following multilevel procedure which describes (for $l = L, \kappa = 0$) one step of our *wavelet frequency decomposition multigrid method* (WFDM):

MLP (l, κ, w, b)
begin
if $l = 0$
then $w := A_{l,\kappa}^{-1} b$
else
$w := S_{l,\kappa}^\nu w + \sum_{j=0}^{\nu-1} S_{l,\kappa}^j L_{l,\kappa}^{-1} b$ (ν steps of the BIM)
$d := A_{l,\kappa} w - b, v_0 = 0, v_1 = 0$
for $j = 1$ **to** γ : **MLP** $(l-1, \kappa 0, v_0, (h \otimes h)d)$
for $j = 1$ **to** γ : **MLP** $(l-1, \kappa 1, v_1, (g \otimes h)\,d)$
$w := w - (h^t \otimes h^t)v_0 - (g^t \otimes h^t)\,v_1$ (5.3)
end

The V-cycle corresponds to the choice $\gamma = 1$ and the W-cycle to $\gamma = 2$.

On any intermediate level $l, 0 \leqslant l \leqslant L, 2^{L-l}$ independent linear systems have to be solved. Hence, the proposed algorithm can be performed in parallel. For a more detailed discussion of this topic, see [12, Sect. 11.3.1].

The consideration of the computational complexity done in [10] carries over to our situation without any modifications. The V-cycle is an $O(N \cdot n_l^2)$-algorithm whereas the W-cycle is of order $O(N \cdot n_l^2 \log n_l)$. If the order N of the wavelet is fixed then the respective orders are $O(n_l^2)$ and $O(n_l^2 \log n_l)$. It is assumed that the connection coefficients are precomputed and the work on level 0 is neglected. Similar assumptions are made in [10].

The precomputation of the connection coefficients $\{\Gamma_\mu^\kappa\}_{\kappa=0,\ldots,2^L}$ requires the solution of a linear system of dimension $4N-3$ for the Γ_μ^0's [14]; the other connection coefficients are generated from these by Mallat transformations involving $O(2^L N^2)$ operations.

5.2. Theoretical Considerations

First, we will show that the two-level methods relative to all intermediate levels of the multigrid scheme are robust; i.e., we verify the robustness of the two-level methods with iteration matrices

$$M_{l,\kappa}(\nu) = \Big[I - (h^t \otimes h^t)A_{l-1,\kappa 0}^{-1}(h \otimes h)A_{l,\kappa} - (g^t \otimes h^t)\,A_{l-1,\kappa 1}^{-1}\,(g \otimes h)\,A_{l,\kappa}\Big] S_{l,\kappa}^\nu, \quad (5.4)$$

where κ is a binary number greater than zero. For $\kappa = 0$ we set $M_{d,0} = M_l$ with M_l from (4.5).

THEOREM 5.1. *Let $M_{l,\kappa}(\nu)$, $\kappa \geqslant 0$, be the iteration matrix defined in* (5.4). *Assume that the BIM $S_{l,\kappa}$ satisfies* (4.3) *and* (4.4) *with A_l replaced by $A_{l,\kappa}$. Then, there exist two positive constants C_κ and σ_κ which are both independent of ε and of the*

$$\|M_{l,\kappa}(\nu)\|_{A_{l,\kappa}} \leqslant C_\kappa \eta(\nu) + \sqrt{r_\kappa(\varepsilon, \delta_l)}, \tag{5.5}$$

where the positive function $r_\kappa(\cdot,\cdot)$ is uniformly bounded by $\sigma_\kappa < 1$. Moreover, $r_\kappa(\varepsilon, \delta_l) \to 0$ for any fixed $\delta_l > 0$ as ε tends to zero.

Remark. The statement of Theorem 5.1 holds correspondingly for the multiplicative version of (5.4); cf. (4.27) and (4.28).

For the proof of the above theorem we need a generalization of Lemma 4.8. The constant σ_κ will be given precisely.

LEMMA 5.2. *Let $\kappa \geqslant 0$ be a binary number. Let $\lambda^\kappa(\xi)$ and $\lambda^{c,\kappa}(\xi)$ be defined by*

$$\lambda^\kappa(\xi) = \sum_\mu \Gamma_\mu^\kappa e^{-i\mu\xi},$$
$$\lambda^{c,\kappa}(\xi) = \sum_\mu \Gamma_\mu^{c,\kappa} e^{-i\mu\xi},$$

with $\Gamma_\mu^{c,\kappa} = \int_{\mathbf{R}} (\psi^{\kappa 0})'(x-\mu)(\psi^{\kappa 1})'(x)dx$. Then,

$$\sigma_\kappa = \sup_{0 \leqslant \xi < 2\pi} \frac{|\lambda^{c,\kappa}(\xi)|^2}{\lambda^{\kappa 0}(\xi)\,\lambda^{\kappa 1}(\xi)} < 1. \tag{5.6}$$

Proof. Similar to Lemma 2.2, we get

$$\lambda^\kappa(\xi) = \sum_\mu \left|(\psi^\kappa)^\wedge(\xi + 2\mu\pi)\right|^2 (\xi + 2\mu\pi)^2, \tag{5.7}$$

$$\lambda^{c,\kappa}(\xi) = \sum_\mu (\psi^{\kappa 0})^\wedge(\xi + 2\mu\pi) \times \overline{(\psi^{\kappa 1})^\wedge(\xi + 2\mu\pi)}\,(\xi + 2\mu\pi)^2. \tag{5.8}$$

If we apply the Fourier transform on both sides of (2.25) and (2.26), then we have $(\psi^{\kappa j})^\wedge(\xi) = m_j(\xi/2)(\psi^\kappa)^\wedge(\xi/2)$, $j = 0, 1$, with m_0 and m_1 from (2.2) and (2.6). Substituting these relations into (5.7) as well as into (5.8) and summing separately over even and odd indices, we arrive at (cf. [1])

$$\lambda^{\kappa 0}(\xi) = 4\left(|m_0(\xi/2)|^2 \lambda^\kappa(\xi/2) + |m_1(\xi/2)|^2 \lambda^\kappa(\xi/2 + \pi)\right), \tag{5.9}$$

$$\lambda^{\kappa 1}(\xi) = 4\left(|m_1(\xi/2)|^2 \lambda^\kappa(\xi/2) + |m_0(\xi/2)|^2 \lambda^\kappa(\xi/2 + \pi)\right), \tag{5.10}$$

$$\lambda^{c,\kappa}(\xi) = 4 m_0(\xi/2)\,\overline{m_1(\xi/2)}\,(\lambda^\kappa(\xi/2) - \lambda^\kappa(\xi/2 + \pi)).$$

Since (2.3), we can rewrite the right-hand side of (5.6) as

$$\sigma_\kappa = \sup_{0 \leqslant \xi < 2\pi} \frac{R^\kappa(\xi)}{R^\kappa(\xi) + \lambda^\kappa(\xi/2)\,\lambda^\kappa(\xi/2 + \pi)}, \tag{5.11}$$

with $R^\kappa(\xi) = |m_0(\xi/2)|^2 |m_1(\xi/2)|^2 (\lambda^\kappa(\xi/2) - \lambda^\kappa(\xi/2+\pi))^2$. Because $\lambda^1(\xi) = \lambda^g(\xi) > 0$ (Lemma 2.3 (iv)) and because m_0 and m_1 are not zero at the same point, it follows inductively from both Eqs. (5.9), (5.10) that $\lambda^\kappa(\xi) > 0$ for all $\kappa \geqslant 1$. This implies $\sigma_\kappa < 1$ for $\kappa > 0$. In the case $\kappa = 0$ assume that $\sigma_0 = 1$. Then, (5.11) implies $\lambda^\kappa(\xi^0/2)\lambda^\kappa(\xi^0/2 + \pi) = 0$ for one $\xi^0 \in [0, 2\pi)$ which in turn, by Lemma 2.3 (ii), implies $|m_0(\xi^0/2)|^2 |m_1(\xi^0/2)|^2 = 0$ which is zero of higher order; see Lemma 2.1 (ii) and (2.4), (2.6). Thus, $\sigma_0 < 1$. ∎

Remark. The numbers σ_κ are again related to strong Cauchy inequalities: if σ is replaced by σ_κ, then (4.30) holds for $v \in W_l^{\kappa 0}$ and $w \in W_l^{\kappa 1}$.

Proof of Theorem 5.1. The proof is similar to the proof of Theorem 4.1. We sketch the steps.

1. The eigenvalues of $A_{l,\kappa}$ are $\Lambda_{\alpha,\beta} = \delta_l^{-2}(\varepsilon\lambda^\kappa(2\pi\alpha/n_l) + \lambda^h(2\pi\beta/n_l)) + 1, 0 \leqslant \alpha, \beta \leqslant n_l - 1$. Therefore, the Lemmata 4.3, 4.6, Corollary 4.5, and Theorem 4.7 hold correspondingly.

2. Let $\tilde{M}_{l,\kappa}$ be defined analogously to (4.21) with $A_{l,\kappa}$ instead of A_l. Then,

$$\|\tilde{M}_{l,\kappa}(\nu) - M_{l,\kappa}(\nu)\|_{A_{l,\kappa}} \leqslant \sigma_\kappa$$

with σ_κ from (5.6). ∎

As a further step in the direction to a robustness proof for the WFDM we will show the uniform boundedness of the constants $\{C_\kappa\}_{\kappa \geqslant 0}$ (5.5). We assume that the constants C_s in (4.3)—where A_l is replaced by $A_{l,\kappa}$—does not depend on κ. This assumption is reasonable because the damped Richardson as well as the damped Jacobi iteration posses this property, if the damping factor is suitably chosen for each κ. For instance, due to the circulant structure of $A_{l,\kappa}$, $\Gamma_0^\kappa / \sum_m |\Gamma_m^\kappa|$ is such a suitable choice for the damping factor of the Jacobi method. We used this choice in our numerical experiments presented in Figs. 3 and 4.

If A_l is replaced by $A_{l,\kappa}$, then the constant in the estimate (4.19) may depend on κ. Therefore, we make a more careful analysis to overcome this difficulty.

LEMMA 5.3. *Let U be defined as in* (4.7) *and let κ be a non-negative binary number. Then,*

$$\left\|A_{l,\kappa}^{1/2}(I - U^t U) A_{l,\kappa}^{-1}\right\| \leqslant C\delta_l,$$

for some constant C which is independent of δ_l, ε, and κ.

Proof. We only give an outline of the proof guided by the steps of the proof of Lemma 4.3. We write n instead of n_l.

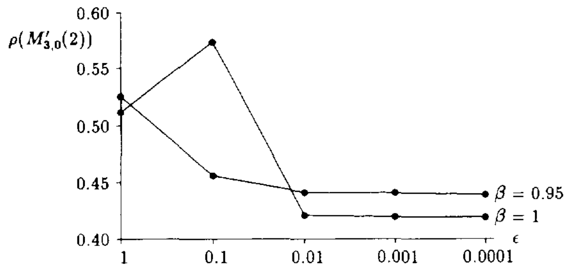

FIG. 3. V-cycle convergence rates of the damped ($\beta = 0.95$) and undamped ($\beta = 1$) WFDM with $L = 3, n_L = 2^L s, s = 16$, and Daubechies wavelet $N = 3$. The damped Jacobi iteration was used as BIM.

1. With Q defined as in (4.9) and with $v_{\alpha,\beta} = v_\alpha \otimes v_\beta$ (3.9) we have

$$\left\|A_{l,\kappa}^{1/2}Q^tQA_{l,\kappa}^{-1}f\right\|^2 = \sum_{\alpha,\beta=0}^{n-1}\sum_{\mu,\nu=0}^{n-1} \Lambda_{\alpha,\beta}^{-1}\Lambda_{\mu,\nu}^{-1}\langle f, v_{\alpha,\beta}\rangle\langle f,\bar v_{\mu,\nu}\rangle T(\alpha,\beta,\mu,\nu)$$

where ($p := h\otimes g, q := g\otimes g$)

$$\begin{aligned}T(\alpha,\beta,\mu,\nu) &= \left\langle A_{l,\kappa}Q^tQv_{\alpha,\beta}, Q^tQ\bar v_{\mu,\nu}\right\rangle \\ &= \underbrace{\left\langle A_{l,\kappa}p^tpv_{\alpha,\beta}, p^tp\bar v_{\mu,\nu}\right\rangle}_{=P_0} + \underbrace{\left\langle A_{l,\kappa}q^tqv_{\alpha,\beta}, q^tq\bar v_{\mu,\nu}\right\rangle}_{=P_1} \\ &\quad + 2\underbrace{\left\langle A_{l,\kappa}p^tpv_{\alpha,\beta}, q^tq\bar v_{\mu,\nu}\right\rangle}_{=P_2} + \underbrace{\left\langle A_{l,\kappa}q^tqv_{\alpha,\beta}, p^tp\bar v_{\mu,\nu}\right\rangle}_{=P_3}.\end{aligned}$$

2. The scalar products P_0, P_1, P_2, and P_3 can be calculated to be ($S_n^{\alpha,\mu}$ is the sum (4.13)):

$$\begin{aligned}P_j &= m_j(2\pi\alpha/n)m_1(2\pi\beta/n) \\ &\quad\times\left(\delta_{l-1}^{-2}\left(\varepsilon\lambda^{\kappa j}(4\pi\alpha/n) + \lambda^g(4\pi\beta/n)\right) + 1\right) \\ &\quad\times m_j(2\pi\mu/n)\, m_1(2\pi\nu/n)S_n^{\alpha,\mu}S_n^{\beta,\nu}, \quad j = 0,1, \\ P_2 &= m_0(2\pi\alpha/n)m_1(2\pi\beta/n)\left(\delta_{l-1}^{-2}\varepsilon\overline{\lambda^{c,\kappa}(4\pi\alpha/n)}\right) \\ &\quad\times m_1(2\pi\mu/n)\, m_1(2\pi\nu/n)S_n^{\alpha,\mu}S_n^{\beta,\nu}, \\ P_3 &= m_1(2\pi\alpha/n)m_1(2\pi\beta/n)\left(\delta_{l-1}^{-2}\varepsilon\lambda^{c,\kappa}(4\pi\alpha/n)\right) \\ &\quad\times m_0(2\pi\mu/n)\, m_1(2\pi\nu/n)S_n^{\alpha,\mu}S_n^{\beta,\nu}.\end{aligned}$$

3. For $j = 0,1$, we have

$$\sum_{\alpha,\beta=0}^{n-1}\sum_{\mu,\nu=0}^{n-1} P_j \leqslant C\delta_l^2\|f\|^2 \times \max\frac{|m_1(2\pi\beta/n)|\,|m_1(2\pi\nu/n)| \times\left(\varepsilon\lambda^{\kappa j}(4\pi\alpha/n) + \lambda^g(4\pi\beta/n) + 1\right)}{\left(\varepsilon\lambda^\kappa(2\pi\alpha/n) + \lambda^h(2\pi\beta/n)\right)\times\left(\varepsilon\lambda^\kappa(2\pi\mu/n) + \lambda^h(2\pi\nu/n)\right)},$$

where the constant C does not depend on κ. In two steps we show that the maximum is bounded in ε and κ. The Lemma 4.4 yields

$$\frac{|m_1(2\pi\nu/n)|}{\varepsilon\lambda^\kappa(2\pi\mu/n) + \lambda^h(2\pi\nu/n)} \leqslant \max_\xi\frac{|m_1(\xi)|}{\lambda^h(\xi)} < \infty.$$

It remains to estimate

$$R = \frac{|m_1(2\pi\beta/n)|\left(\varepsilon\lambda^{\kappa j}(4\pi\alpha/n) + \lambda^g(4\pi\beta/n) + 1\right)}{\varepsilon\lambda^\kappa(2\pi\alpha/n) + \lambda^h(2\pi\beta/n)}.$$

We use $(a+b)/(c+d) \leqslant \max\{a/c, b/d\}$ for $a,b,c,d > 0$ to get

$$R \leqslant \max\left\{\underbrace{\frac{\lambda^{\kappa j}(4\pi\alpha/n)}{\lambda^\kappa(2\pi\alpha/n)}}_{=R_1}, \underbrace{\frac{|m_1(2\pi\beta/n)|\left(\lambda^g(4\pi\beta/n) + 1\right)}{\lambda^h(2\pi\beta/n)}}_{=R_2}\right\}.$$

The quotient R_2 is bounded independent of κ due to (4.17). To bound R_1 we consider

$$\frac{\lambda^{\kappa j}(2\xi)}{\lambda^\kappa(\xi)} \leqslant 4\left(1 + \frac{\lambda^\kappa(\xi+\pi)}{\lambda^\kappa(\xi)}\right),$$

where the inequality follows by (5.9) and (5.10), respectively. Since,

$$\frac{\max_\xi\lambda^\kappa(\xi)}{\min_\xi\lambda^\kappa(\xi)} \leqslant \frac{\max_\xi\lambda^g(\xi)}{\min_\xi\lambda^g(\xi)} \quad \text{for all } \kappa \geqslant 1,$$

which also follows from (5.9) and (5.10), R_1 is uniformly bounded in κ.

If we estimate $\sum_{\alpha,\beta}\sum_{\mu,\nu}(P_2 + P_3)$ in the same manner using $|\lambda^{c,\kappa}(\xi)|^2 \leqslant \lambda^{\kappa 0}(\xi)\lambda^{\kappa 1}(\xi)$, then Lemma 5.3 is proved. ■

COROLLARY 5.4. *If the BIM $S_{l,\kappa}$ satisfies* (4.3) *with a uniform constant C_s in κ, then the constants $\{C_\kappa\}_{\kappa\geqslant 0}$ in* (5.5) *are uniformly bounded: $C_\kappa \leqslant C_b$ for all $\kappa \geqslant 0$.*

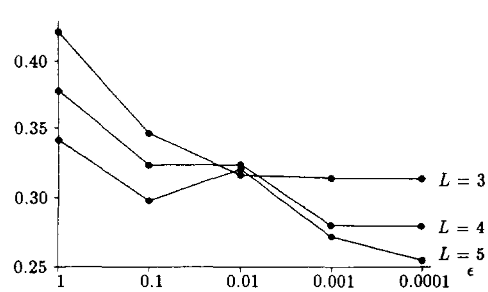

FIG. 4. W-cycle convergence rates of the WFDM with damped Jacobi iteration as BIM, $n_L = 2^L s, s = 16$, and Daubechies wavelet $N = 3$.

Next, we consider the iteration matrix $M'_{L,0}(\nu)$ of the WFDM. Let $M'_{l,\kappa}(\nu)$ denote the iteration matrix of the iteration ($l \geq 1, \kappa \geq 0$)

$$\begin{aligned} w &:= U_l^m, \\ \mathrm{MLP}\,(l, \kappa, w, F_l)&, \\ U_l^{m+1} &:= w; \end{aligned} \tag{5.12}$$

that is, we start our multigrid procedure to solve $A_{l,\kappa}U_l = F_l$ on level l.

LEMMA 5.5. *The above defined matrices $M'_{l,\kappa}(\nu)$ for the iteration* (5.12) *satisfy the recursion*

$$\begin{aligned} M'_{l,\kappa}(\nu) &= M_{l,\kappa}(\nu) + (h' \otimes h')\left(M'_{l-1,\kappa 0}(\nu)\right)^{\gamma} \\ &\quad \times A_{l-1,\kappa 0}^{-1}(h \otimes h) A_{l,\kappa} S_{l,\kappa}^{\nu} + (g' \otimes h')\left(M'_{l-1,\kappa 1}(\nu)\right)^{\gamma} \\ &\quad \times A_{l-1,\kappa 1}^{-1}(g \otimes h) A_{l,\kappa} S_{l,\kappa}^{\nu}, \end{aligned} \tag{5.13}$$

with $M'_{0,\kappa}(\nu) = 0$. The matrices $M_{l,\kappa}(\nu)$ are defined in (5.4). *If the BIM $S_{l,\kappa}$ satisfies* (4.4), *then*

$$\begin{aligned} \|M'_{l,\kappa}(\nu)\|_{A_{l,\kappa}} &\leq \|M_{l,\kappa}(\nu)\|_{A_{l,\kappa}} + \|M'_{l-1,\kappa 0}(\nu)\|^{\gamma}_{A_{l-1,\kappa 0}} \\ &\quad + \|M'_{l-1,\kappa 1}(\nu)\|^{\gamma}_{A_{l-1,\kappa 1}}. \end{aligned} \tag{5.14}$$

Proof. The recursion (5.13) follows by transfering the proof of Lemma 7.1.4 in [9] to the additive coarse grid correction. For the proof of (5.14) we remark:

$$\begin{aligned} &\left\|A_{l,\kappa}^{1/2}(h' \otimes h') A_{l-1,\kappa 0}^{-1/2}\right\|^2 \\ &\qquad = \rho\left((h \otimes h) A_{l,\kappa}(h' \otimes h') A_{l-1,\kappa 0}^{-1}\right) = 1, \end{aligned}$$

where the last equality holds since $(h \otimes h)A_{l,\kappa}(h' \otimes h') = A_{l-1,\kappa 0}$. ∎

A robustness proof for the V-cycle fails with the usual techniques. Indeed, numerical experiments (see, e.g., Fig. 3 ($\beta = 1$)), raise doubts on the robustness. The convergence rate behaves like that one of the V-cycle of the usual multigrid method (based on Finite Element or Finite Difference discretization) with ILU as BIM; see [26]. In the latter case robustness had been achieved by damping the ILU method. Here we produce the same effect, if we damp the additive coarse grid correction, that is, we replace (5.3) in the algorithmic representation of MLP by ($\beta \in \mathbf{R}$)

$$w := w - \beta\,(h' \otimes h')\,v_0 - \beta\,(g' \otimes h')\,v_1.$$

With a right choice of the damping factor β, the V-cycle seems to be robust; see Fig. 3 ($\beta = 0.95$). For an analysis of the damped V-cycle we refer to [21].

For the W-cycle the standard techniques, see [9, Chap. 7], may be successful. The idea is to analyze the sequence

$$\xi_{l,\kappa_l} = \zeta_{\kappa_l} + \xi^2_{l-1,\kappa_l 0} + \xi^2_{l-1,\kappa_l 1} \tag{5.15}$$

with $1 \leq l \leq L, 0 \leq \kappa_l \leq 2^{L-1} - 1$($\kappa_l$ binary), $\xi_{0,\kappa} = 0$ and $\zeta_\kappa = C_b\eta(\nu) + \sqrt{\sigma_\kappa}$. The connection of (5.15) with (5.14) is obvious.

Since $\zeta_\kappa > \sqrt{\sigma_\kappa}$, the ζ_κ's can not be made arbitrarily small by just increasing ν. Therefore, the convergence of the W-cycle does not follow directly from the two-level convergence as in [9]. However, if the apparent decrease of the σ_κ's (5.6), see Table 1, could be verified analytically, then the above approach would yield the robustness of the W-cycle. A qualitative assertion that the σ_κ's tend to zero will not be sufficient to prove the robustness; the decay rate of such a sequence tending to zero might be necessary.

6. CONCLUSION

In this paper, the robustness of any two-level method corresponding to an intermediate level in the WFDM has been proved (Theorem 5.1). For the complete multilevel process we only presented some analysis toward a robustness proof. More specifically, the quantitative decay of the $\sigma_{\kappa'}$'s is needed to ensure W-cycle convergence. However, in [21] we are able to prove a level dependent convergence of the damped V-cycle.

It is also worth to mention that the formulation of the FDMGM as an additive Schwarz iteration [11] using the wavelet packet subspaces deserves further exploration.

Outlook on More General Boundary-Value Problems

(a) Dirichlet Boundary Conditions in General Domains. A wavelet-Galerkin discretization in combination with a penalty/fictitious domain formulation of the Dirichlet problem

$$\begin{aligned} -\varepsilon u_{xx} - u_{yy} + u = f \quad &\text{in } \omega, \\ u = g \quad &\text{on } \partial\omega, \end{aligned} \tag{5.16}$$

TABLE 1
The Numbers σ_κ for the Daubechies Wavelets of Order 3 and 4

	σ_0	σ_1	σ_{10}	σ_{11}	σ_{100}	σ_{101}	σ_{110}	σ_{111}
$N = 3$	2.9E-1	2.8E-2	2.4E-3	2.3E-2	1.9E-4	1.6E-3	5.5E-3	5.0E-3
$N = 4$	1.9E-1	8.5E-3	6.9E-4	6.3E-3	5.0E-5	5.1E-4	1.3E-3	1.4E-4

where ω is a domain in $\mathbf{R}^2$, yields the following linear system

$$(A_l + \gamma^{-1} M_l)\, U_l^\gamma = F_l + \gamma^{-1} M_l G_l, \tag{5.17}$$

with appropriate right hand-sides F_l and G_l; see [23, 24]. The matrix A_l is the one from (3.6) and M_l is a diagonal matrix with diagonal entries either 0 or 1 (the entries are 1 if the corresponding unknowns belong to the (discrete) boundary and they are 0 otherwise).

It is shown in [8] how the WFDM can be used as a preconditioner for the conjugate gradient iteration to become an efficient solver for (5.17) under the assumption of a (sufficiently) small penalty parameter $\gamma > 0$.

(b) The Case of Space-Dependent ε. To treat a space-dependent $\varepsilon = \varepsilon(x, y)$ in (1.1) (resp. in (5.16)) we project ε onto the space $V_l^p \otimes V_l^p$ (3.3), that is, we make the ansatz $\varepsilon(x, y) = \sum \varepsilon_{i,j}^l \varphi_{l,i}(x) \varphi_{l,j}(y)$. Then, the wavelet-Galerkin discretization leads to the higher order connection coefficients

$$\Gamma_{k,l,p} = \int \varphi(x-k)\, \varphi'(x-l)\, \varphi'(x-p)\, dx,$$

the stable evaluation of which is considered in [14].

The adaptation of the WFDM to this more general case has not been carried out at the present time, but it seems as if it will be necessary to use both $h \otimes g$ and $g \otimes h$ as extra filters in this more complex setting.

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