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Wavelet analysis and the geometry of Euclidean domains

Howard L. Resnikoff

AWARE, Inc., Cambridge, MA, USA

Raymond O. Wells Jr. *

Department of Mathematics, Rice University, USA

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This paper gives an explicit representation for a multiresolution of Euclidean domains and their boundaries in terms of a wavelet system defined in the ambient space. The exterior derivative of the characteristic function of a domain is represented in an infinite series of compactly supported wavelet functions whose supports intersect the geometric boundary. This is used to obtain representations of the boundary integrals which appear in weak solutions of partial differential equations.

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1. Introduction

In this paper we want to discuss a wavelet representation for geometric regions in Euclidean space. Wavelet analysis is a new branch of harmonic analysis with applications to many areas of mathematics, science, and technology. In section 2 we give an overview of wavelet analysis in general. In section 3 we discuss the well-known distribution representation of geometric domains and their boundaries. In section 4 we combine these two points of view to provide explicit wavelet formulas for boundary integrals which appear in most treatments of boundary value problems. We feel that these formulas will help with the numerical solution of boundary value problems, as it will give a common language for representing the interior functions, the boundary data, and the geometry, in a way which can be truncated readily at any desired scale for numerical purposes. The ideas represented work in any number of

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dimensions, but the exposition is in two variables, as that is where the first numerical experiments are being carried out.

The authors would like to dedicate this paper to Roger Penrose, whose geometric ideas in a range of contexts have been an inspiration to several generations of mathematicians and scientists.

2. Compactly supported wavelets

The first orthonormal basis of compactly supported wavelets was introduced by Haar in his 1910 dissertation [1] but Haar's functions, which are piecewise constant and supported on finite intervals, are discontinuous and therefore largely unsuitable for the numerical solution of differential equations. In 1988 Daubechies [2] published the first construction of smooth compactly supported orthonormal bases for $L^2(\mathbf{R})$. The smoothness of Daubechies' wavelets varies inversely with the length of the support of the "largest" basis function. The Haar basis arises as the least smooth – the "shortest" – of the Daubechies bases. The next smoothest case consists of continuous but nowhere differentiable functions the length of whose support is at most 3, and whose graph is a fractal curve.

The Daubechies bases are constructed as follows: A $2 \times 2g$ matrix a of complex numbers is said to be a *wavelet matrix* if it satisfies

$$\sum_k a_k^s = 2\delta^{s,0} , \tag{1}$$

$$\sum_k \overline{a}_{k+2l'}^{s'} a_{k+2l}^{s} = 2\delta^{s',s}\delta_{l',l} . \tag{2}$$

For a given wavelet matrix a a *scaling function* $\varphi[a] : \mathbf{R} \to \mathbf{C}$ is implicitly defined by the "2-scale recursion"

$$\varphi[a](x) = \sum_k a_k^0 \varphi[a](2x-k) . \tag{3}$$

It is known that this recursion has at least one solution, and that it is has exactly one if the "1" eigenvalue of a certain matrix associated with a has multiplicity 1 (cf. Lawton [3]); we shall restrict ourselves to this case.

Define the *basic wavelet* $\psi[a]$ by the formula

$$\psi[a](x) = \sum_k a_k^1 \varphi[a](2x-k) , \tag{4}$$

and introduce the auxiliary functions

$$\varphi_l[a](x) := \varphi[a](x-l)\,, \quad l \in \mathbf{Z}, \tag{5}$$

$$\psi_{j,k}[a](x) := 2^{j/2}\psi[a](2^j x - k)\,. \tag{6}$$

Then

$$\{\psi_{j,k}[a](x) : j,k \in \mathbf{Z}\} \tag{7}$$

is a complete orthonormal basis for $L^2(\mathbf{R})$, and

$$\{\varphi_l[a](x) : l \in \mathbf{Z}\} \cup \{\psi_{j,k}[a](x) : j,k \in \mathbf{Z}, 0 \le j\} \tag{8}$$

is a complete orthonormal basis for a larger function space; cf., e.g., refs. [2–5].

The function $\psi_{j,k}[a](x)$ is a translation of a "scaled" version of $\psi[a](x)$; the length of the support of $\psi_{j,k}[a](x)$ is 2^{-j} times the length of the support of $\psi[a]$. These properties – that the basis functions are translated and scaled versions of one another, and that there are basis functions whose support is arbitrarily small – are two of the four critical properties that make compactly supported wavelets useful for solving nonlinear partial differential equations. The other two properties are low computational workload required to calculate the coefficients of the wavelet expansion of a function, and the ability to control the smoothness of the basis functions.

One recent line of thought in the numerical solution of partial differential equations exploits the efficiency of hierarchical methods for computing. This so-called "multigrid" approach (cf. ref. [6]) derives its advantages from the same source as does positional notation for representing numbers, but it does not address problems related to the fine structures usually associated with nonlinear equations. Because of their multiscale structure, which is implicit in the recursion that defines the scaling function, wavelets share with multigrid the ability to economize computation by adjusting the quantity of computation to the "size" of the error terms that are being reduced at each stage of the solution.

The translated scaling functions $\{\varphi_l[a](x) : l \in \mathbf{Z}\}$ span a subspace of functions. In many respects they are similar in their approximation and representation properties to finite elements, or to splines (cf. ref. [7]). Like finite elements and splines, the scaling functions are a partition of unity, i.e.,

$$\sum_l \varphi[a](x-l) = 1\,.$$

They have the advantage of orthonormality, which implies the minor disadvantage that the scaling function is not nonnegative. Much more important is that the set of translated wavelet scaling functions is completed to an orthonormal basis by the scaled wavelets, whereas there is no natural intrinsic way to complete a set of finite elements or spline functions. Investigation of wavelets with "higher multiplier" (replace the scale "2" by a larger number) shows that there is a deeper connection between wavelets and splines. Roughly put, in the limit of "infinite multiplier", the wavelet basis splits into a "low-pass" orthonormal space spanned by orthonormal piecewise polynomial functions, and a "high-pass" space spanned by infinite multiplier wavelet functions. In one special case, for instance, one recovers the theory of Fourier series in a novel setting. Our investigations of this topic will appear elsewhere.

The finite element method is probably the most practically successful "standard" means for numerically solving partial differential equations. The variational formulation of Galerkin is particularly amenable to modification for a wavelet solution. This "Wavelet–Galerkin" approach has achieved promising results in its initial application to the numerical solution of periodic (i.e., problems without boundary conditions.) One- and two-dimensional nonlinear problems including Burgers' equation [8,9], the gas dynamics equation [10], and two dimensional turbulence [11,12]. In the latter case, an extensive investigation has shown that the Wavelet–Galerkin method has superior stability and lower computational workload when compared with a standard de-aliased spectral solution method. The evidence of the ability of wavelets to stably capture fine scale nonlinear phenomena encourages extension of the method to more complex and realistic problems, and, in particular, to the investigation of the interaction of the wavelet representation with the boundary conditions for a problem, which is the motivation for the present note.

Every application of wavelets to the solution of differential equations requires that derivatives of functions be directly expressible in terms of the coefficients of the wavelet expansion of the function itself. In this regard, nothing could be simpler than the situation for the Fourier series basis, where the derivative of a basis function is proportional to the basis function. But nothing this simple could occur for compactly supported wavelets, for it is easy to show that a scaling function (and hence, a wavelet) can have only a finite order of differentiability. So we must expect a more complicated formulation. The coefficients of the wavelet expansion of the derivatives of $\varphi[a](x)$, when they exist, are called *connection coefficients*. It turns out that the connection coefficients satisfy numerous algebraic identities, which simplifies their calculation, and they lead to rapidly convergent wavelet series expansions. The details, which appear to be essential for the efficient use of wavelets for the solution of differential equations, are described in ref. [13]. Their use is described in the representation of geometric regions in section 3.

3. Geometry and distributions

Let D be an open connected set in $\mathbf{R}^2$ which is bounded and which has a boundary ∂D which is piecewise smooth with a finite number of singularities, and which is rectifiable. We call such a domain *admissible*. Let

$$\chi_D(x) := \begin{cases} 1, & x \in D, \\ 0, & x \notin D, \end{cases}$$

be the characteristic function of the domain. We shall see that χ_D has all of the geometric information about the boundary of D, and we want to express this in distribution terms.

Let $\mathcal{E}^p(\mathbf{R}^2)$ denote the C^∞ differential p-forms in $\mathbf{R}^2$. That is:

$$\begin{aligned} \mathcal{E}^0(\mathbf{R}^2) &= C^\infty \text{ function in } \mathbf{R}^2 = C^\infty(\mathbf{R}^2)\,, \\ \mathcal{E}^1(\mathbf{R}^2) &= \{f\,\mathrm{d}x + g\,\mathrm{d}y : f, g \in C^\infty(\mathbf{R}^2)\}\,, \\ \mathcal{E}^2(\mathbf{R}^2) &= \{f\,\mathrm{d}x \wedge \mathrm{d}y : f \in C^\infty(\mathbf{R}^2)\}\,. \end{aligned}$$

The mapping of exterior differentiation

$$\mathcal{E}^0(\mathbf{R}^2) \xrightarrow{d} \mathcal{E}^1(\mathbf{R}^2) \xrightarrow{d} \mathcal{E}^2(\mathbf{R}^2)$$

is defined by

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y,$$

for $f \in \mathcal{E}^0(\mathbf{R}^2)$, and by

$$\mathrm{d}\omega = \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y,$$

for $\omega \in \mathcal{E}^1(\mathbf{R}^2)$. We define the *dual* of a one-form ω by

$$*\omega = -g\,\mathrm{d}x + f\,\mathrm{d}y$$

so that

$$\begin{aligned} \omega \wedge *\omega &= (f\,\mathrm{d}x + g\,\mathrm{d}y) \wedge (-g\,\mathrm{d}x + f\,\mathrm{d}y) \\ &= (f^2 + g^2)\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Hence

$$\int_{\mathbf{R}^2} \omega \wedge *\omega = \|\omega\|^2_{L^2}$$

is an L^2-norm for ω viewed as a vector-valued function in $\mathbf{R}^2$.

The spaces $\mathcal{E}^p(\mathbf{R}^2)$ have the topology of uniform convergence of all derivatives on compact sets. Define the topological dual spaces

$$\mathcal{K}^{2-p}(\mathbf{R}^2), \quad p = 0, 1, 2.$$

We want to characterize $\mathcal{K}^r(\mathbf{R}^2)$ more closely. First, we note that

$$\mathcal{K}^0(\mathbf{R}^2) \cong \mathcal{E}'(\mathbf{R}^2),$$

where $\mathcal{E}'(\mathbf{R}^2)$ consists of the distributions on $\mathbf{R}^2$ with compact support. Moreover

$$\mathcal{K}^1(\mathbf{R}^2) \cong \left\{T_1\,\mathrm{d}x + T_2\,\mathrm{d}y : T_j \in \mathcal{E}'(\mathbf{R}^2)\right\},$$
$$\mathcal{K}^2(\mathbf{R}^2) \cong \left\{T\,\mathrm{d}x \wedge \mathrm{d}y : T \in \mathcal{E}'(\mathbf{R}^2)\right\}.$$

That is, $\mathcal{K}^p(\mathbf{R}^2)$ consists of differential p-forms with distribution coefficients with compact support. Let us indicate the duality pairings which are involved. For $T \in \mathcal{K}^0$, $\omega \in \mathcal{E}^2$, we have, letting $\omega = \varphi\,\mathrm{d}x \wedge \mathrm{d}y$, $\varphi \in \mathcal{E}(\mathbf{R}^2)$,

$$\langle T, \omega\rangle = \int T\varphi\,\mathrm{d}x \wedge \mathrm{d}y$$

in the case where T is smooth, and more generally,

$$\langle T, \omega\rangle := \langle T, \varphi\rangle.$$

If

$$T = T_1\,\mathrm{d}x + T_2\,\mathrm{d}y \in \mathcal{K}^1(\mathbf{R}^2),$$

and $\omega = \varphi_1\,\mathrm{d}x + \varphi_2\,\mathrm{d}y$, is a C^∞ one-form, then we define

$$\langle T, \omega\rangle = \langle T_1, \varphi_2\rangle - \langle T_2, \varphi_1\rangle. \tag{9}$$

If T were a smooth one-form then this would correspond to

$$\begin{aligned}\langle T, \omega\rangle &= \int T \wedge \omega \\ &= \int (T_1\,\mathrm{d}x + T_2\,\mathrm{d}y) \wedge (\varphi_1\,\mathrm{d}x + \varphi_2\,\mathrm{d}y) \\ &= \int (T_1\varphi_2 - T_2\varphi_1)\,\mathrm{d}x \wedge \mathrm{d}y \\ &= \int T_1\varphi_2\,\mathrm{d}x \wedge \mathrm{d}y - \int T_2\varphi_1\,\mathrm{d}x \wedge \mathrm{d}y \\ &= \langle T_1, \varphi_2\rangle - \langle T_2\varphi_1\rangle,\end{aligned}$$

which is the origin of formula (9) above. The duality between $\mathcal{K}^2$ and $\mathcal{E}^0$ is just like the first one.

Theorem 1. *If D is an admissible domain, then $\mathrm{d}\chi_D \in \mathcal{K}^1(\mathbf{R}^2)$, and moreover*

$$\operatorname{supp} \mathrm{d}\chi_D = \partial D. \qquad \square$$

We now define, for $\omega \in \mathcal{E}^1(\mathbf{R}^2)$,

$$\int_{\partial D} \omega := \langle \omega, \mathrm{d}\chi_D \rangle, \tag{10}$$

which is well defined by the theorem. Moreover, the left hand side is well defined for any form $\omega \in \mathcal{E}^1(U)$, where U is a neighborhood of ∂D. One can check that this definition on (10) is equivalent to the usual one, and indeed we have

$$\int_{\partial D} \omega = \langle \omega, \mathrm{d}\chi_D \rangle = \langle \chi_D, \mathrm{d}\omega \rangle = \int_D \mathrm{d}\omega,$$

which is Stokes' formula. The proof of the above theorem is standard (see, e.g., ref. [14]), and will be omitted.

4. Geometry and wavelets

Let D be an admissible domain in $\mathbf{R}^2$ and let χ_D be the characteristic function for D. Let $\{\psi_{j\alpha}(x)\}$ be an orthonormal wavelet system in $\mathbf{R}^2$ indexed by $\alpha \in \mathbf{Z}^2$, a lattice in $\mathbf{R}^2$, and by $j \in \mathbf{Z}^+$, the nonnegative integers. More specifically, $\psi_{j\alpha}$ can be tensor products of one-dimensional wavelet functions. The notation also hides the fact that $\psi_{j\alpha}$ might consist of scaling and wavelet functions. But for simplicity, we will simply write $\psi_{j\alpha}$, noting that supp $\psi_{j\alpha}$ depends on the location of the lattice point, and the index j denotes the *level* of the wavelet. In addition we will have

$$\operatorname{diam}(\operatorname{supp} \psi_{j\alpha}) = \mathrm{O}(2^{-j}).$$

The support set for tensor products will be rectangles, and for some irreducible wavelets they will be domains with fractal type boundaries.

We can expand the characteristic function of D in this wavelet basis, obtaining

$$\chi_D(x) = \sum_{j\alpha} c^{j\alpha} \psi_{j\alpha}(\alpha).$$

Now we can differentiate, obtaining

$$\mathrm{d}\chi_D(x) = \sum_{j\alpha} c^{j\alpha} \left(\frac{\partial \psi_{j\alpha}}{\partial x} \mathrm{d}x + \frac{\partial \psi_{j\alpha}}{\partial y} \mathrm{d}y \right). \tag{11}$$

We now introduce the *connection coefficients*. Define

$$\frac{\partial \psi_{j\alpha}}{\partial x} = \sum_{l\beta} \rho_{j\alpha}^{l\beta} \psi_{l\beta}, \qquad \frac{\partial \psi_{j\alpha}}{\partial y} = \sum_{l\beta} \sigma_{j\alpha}^{l\beta} \psi_{l\beta}, \tag{12}$$

where the connection coefficients $\{\rho_{j\alpha}^{l\beta}\}$ correspond to the x-direction and $\{\sigma_{j\alpha}^{l\beta}\}$ correspond to the y-direction. Since we can assume that $\psi_{j\alpha} \in C^1(\mathbf{R}^2)$, it follows that the series in (12) converges uniformly on compact subsets of $\mathbf{R}^2$. Replacing these series in (11), we find that

$$\begin{aligned} \mathrm{d}\chi_D(x) &= \Big(\sum_{j\alpha,l\beta} c^{j\alpha} \rho_{j\alpha}^{l\beta}\Big)\mathrm{d}x + \Big(\sum_{j\alpha,l\beta} c^{j\alpha} \sigma_{j\alpha}^{l\beta} \psi_{l\beta}\Big)\mathrm{d}y \\ &= K_1\,\mathrm{d}x + K_2\,\mathrm{d}y, \end{aligned} \tag{13}$$

where $K_1, K_2 \in \mathcal{E}^1(\mathbf{R}^2)$, and have the series representation given in (13).

Suppose that $\omega = \omega_1\,\mathrm{d}x + \omega_2\,\mathrm{d}y$ is a smooth one-form in $\mathbf{R}^2$, which we can assume vanishes outside some neighborhood of ∂D. Then

$$\omega = \omega_1^{j\alpha} \psi_{j\alpha}\,\mathrm{d}x + \omega_2^{j\alpha} \psi_{j\alpha}\,\mathrm{d}y,$$

using the Einstein summation convention. It follows that

$$\begin{aligned} \int_{\partial D} \omega &= \langle \omega, \mathrm{d}\chi_D \rangle \\ &= \langle \omega_1\,\mathrm{d}x + \omega_2\,\mathrm{d}y, K_1\,\mathrm{d}x + K_2\,\mathrm{d}y \rangle \\ &= \langle K_2, \omega_1 \rangle - \langle K_1, \omega_2 \rangle. \end{aligned}$$

If a distribution with compact support has the form $T = T^{j\alpha}\psi_{j\alpha}$, and $\varphi = \varphi^{j\alpha}\psi_{j\alpha}$ is a smooth function (test function), then

$$\langle T, \varphi \rangle = \sum_{j\alpha} T^{j\alpha} \varphi^{j\alpha}$$

is a representation for the duality pairing between T and φ. This follows from the orthonormality of the basis $\{\psi_{j\alpha}\}$. Thus we have the theorem

Theorem 2. *Let D be an admissible domain in $\mathbf{R}^2$, then*

$$\int_{\partial D} \omega = \sum_{j\alpha,l\beta} c^{j\alpha} (\sigma_{j\alpha}^{l\beta} \omega_2^{l\beta} - \rho_{j\alpha}^{l\beta} \omega_1^{l\beta}). \tag{14}$$

Proof. We see that

$$\begin{aligned}\int_{\partial D} \omega &= \langle \mathrm{d}\chi_D, \omega \rangle \\ &= \langle K_1, \omega_2 \rangle - \langle K_2, \omega_1 \rangle \\ &= \sum_{j\alpha, l\beta} (c^{j\alpha} \sigma_{j\alpha}^{l\beta} \omega_2^{l\beta} - c^{j\alpha} \rho_{j\alpha}^{l\beta} \omega_1^{l\beta}) \\ &= \sum_{j\alpha, l\beta} c^{j\alpha} (\sigma_{j\alpha}^{l\beta} \omega_2^{l\beta} - \rho_{j\alpha}^{l\beta} \omega_1^{l\beta}). \end{aligned}$$ □

We note that in the sums

$$\begin{aligned} K_1 &= \sum_{j\alpha, l\beta} (c^{j\alpha} \sigma_{j\alpha}^{l\beta}) \psi_{l\beta}, \\ K_2 &= \sum_{j\alpha, l\beta} (c^{j\alpha} \rho_{j\alpha}^{l\beta}) \psi_{l\beta}, \end{aligned}$$

the sums only go over those wavelets $\psi_{l\beta}$ such that supp $\psi_{l\beta} \cap \partial D \neq \emptyset$, and this depends only on the geometry of ∂D and the support of the wavelet systems. We see that

$$\int_{\partial D} \omega = \sum_{j=0}^{\infty} \left(\sum_{j\alpha, l\beta} c^{j\alpha} \left(\sigma_{j\alpha}^{l\beta} \omega_{2_2}^{l\beta} - \rho_{j\alpha}^{l\beta} \omega_1^{l\beta} \right) \right)$$

provides a multiresolution of the boundary integral. By truncating the data in a suitable fashion we can consider finite sum approximations to this integral, which could help in numerical work. A number of examples of this have been worked out, and will appear in forthcoming publications.

There are a number of points in this analysis which need to be verified in greater detail, but this outline is to stimulate further study and numerical experimentation.

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