Nonlinear Field Equations and Twistor Theory

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1. Introduction

In 1967 Roger Penrose introduced what is now called *twistor geometry* (Penrose 1967). This geometry is an amplification of the geometry of lines in projective space studied extensively by Felix Klein and others in the late 19th century. Penrose made the fundamental observation that this classical geometry could be used in conjunction with a Radon transform style integral geometry to provide rather complete sets of solutions of numerous partial differential equations, both linear and nonlinear, which arise naturally in theoretical physics. The Radon transform, which involves integration over straight lines in $\mathbf{R}^3$, was used by F. John to generate solutions of hyperbolic partial differential equations in 1942. Penrose integrates over complex projective lines, real 2-spheres, and the data of the integration is complex-analytic functions of 3 complex variables. This is analogous to (and indeed related to) Weierstrass' representation of solutions of the minimal surface equation in terms of triples of holomorphic functions of a single complex variable which satisfy an algebraic relation.

In this paper we discuss the origins of the differential equations which are amiable to twistor-geometric study from the realms of theoretical physics. Twistor geometry is most suitable for physical theories which are relativistically invariant, including Maxwell's description of electromagnetic phenomena from the 19th century (Maxwell's equations), Einstein's theory of gravitation (the equations of general relativity), and the contemporary descriptions of elementary particles (quantum field theory). Twistor geometry interacts in a nontrivial manner with the mathematical models involved in all of the above physical theories. In particular, it has led to new solutions of some of the nonlinear partial differential equations involved (Einstein's equations, Yang-Mills equations, equations of a magnetic monopole).

We will describe some recent progress on twistor-geometric representations of solutions of classical field equations. There are two aspects to this. First, there is the problem of translating a given system of differential equations into twistor-geometric language. This has been done successfully in many different cases, and will be described in Section 3. Once a given problem has been transferred to a twistor-geometric context, it becomes, in general, a specific problem involving algebraic geometry, several complex variables, and algebraic topology involving holomorphic data on complex manifolds. The only partial differential equations remaining in the twistor context are the Cauchy-Riemann equations characterizing holomorphic functions. In effect, the partial differential equation has been transformed into a geometric problem. This is analogous to the Fourier transform transforming a partial differential equation with constant coefficients into an algebraic problem, where differentiation is replaced by the (algebraic) symbol of the differential operator in question. The second aspect of the recent progress on the twistor-geometric representation of the solutions of the classical field equations concerns itself with solving the geometric problem in question and transforming back to space-time (or Euclidean space), generating (sometimes new) solutions of the classical field equations.

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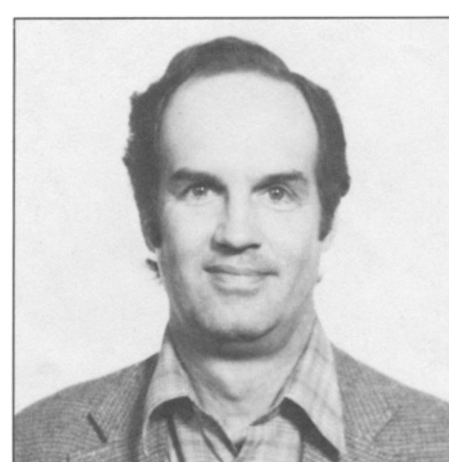

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In Section 2 we discuss Minkowski and Euclidean 4-dimensional space and their natural transformation groups. The field theories which are of interest here arise from calculus of variation problems in a natural way, just as geodesics arise from the calculus of variations in the specific case of the arc length integral. These ideas are discussed in the context of contemporary physics. In Section 3 we discuss the fundamentals of twistor geometry, and indicate the nature of the integral-geometric transform which we now call the *Penrose transform.* This transform is at once reminiscent of both the Radon transform and the Fourier transform and has features of both. It generates automatically solutions to various field equations, and is discussed in both a general and a concrete manner.

2. Minkowski Space and Euclidean Space

Let M^4 be real affine Minkowski space, i.e., $M^4 = \mathbf{R}^4$ equipped with a pseudometric

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad 2.1$$

This space was introduced by Minkowski shortly after Einstein postulated his special theory of relativity. The fundamental physical premise is that any law of physics should be invariant under transformations leaving the pseudometric (2.1) invariant (generalization of Euclidean motion). We call the group of affine transformations which preserve (2.1) the *Poincaré group* (Poincaré was one of the inventors of special relativity also!).

This space forms the background space for most of modern physics. The *light rays* are the lines in M whose tangent vectors are null; that is, they satisfy $\|v\|_{M^4} = 0$ with respect to the metric form above. The light rays passing through a given point p form the light cone C_p at the point p.

We can parametrize the light rays in M by considering a family of light cones C_p, where p varies on a space-like hyperplane, for example, $H = \{x^0 = 0\} \cong \mathbf{R}^3$. The different light rays in C_p for a fixed p are parametrized by a 2-sphere (the "celestial sphere" of an observer). Thus we see that we have

$$\{\text{light rays in } M^4\} \cong \mathbf{R}^3 \times S^2, \qquad 2.2$$

a 5-dimensional real manifold. We can identify S^2 with $\mathbf{P}_1(\mathbf{C})$, the complex projective line, and thus introduce partially complex coordinates into the space of light rays. We can identify a point in M^4 with the light rays passing through that point which will correspond to a specific 2-sphere in $\mathbf{R}^3 \times S^2$ under the bijection (2.2). This is part of the twistor correspondence that we discuss in more detail below. Penrose's basic principle is to transform problems in M^4 to problems on $\mathbf{R}^3 \times S^2$ using the above correspondence. To see this more clearly, we need more structure, which we describe in Section 3 below.

Euclidean space E^4 is simply $\mathbf{R}^4$ with the positive definite metric $ds^2 = (dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2$. In both Minkowski space and Euclidean space we do not have a distinguished origin, hence these are *affine* spaces, not *vector* spaces. But we can always choose coordinates with respect to some origin for calculational purposes. To transform from M^4 to E^4 with respect to standard coordinate systems we simply have the mapping

$$\begin{aligned}(x^0,x^1,x^2,x^3) \to &- (ix^0,x^1,x^2,x^3)\\ &= (y^0,y^1,y^2,y^3) \in \mathbf{E}^4,\end{aligned} \qquad 2.3$$

that is, we multiply the "time coordinate" x by $i = \sqrt{-1}$ (and then change signs to get positive definiteness). This transformation is important in the physicists' "Euclidization" of quantum field theory. By rotating coordinates, certain exponentially oscillatory asymptotic integrals become damped exponential integrals, and hence are much more convergent.

The *conformal group* was introduced into physics when it was noticed early in this century that Maxwell's equations are invariant not only under the Poincaré group but also under additional dilations and certain types of inversions (analogous to the conformal group of Möbius transformations in the complex plane). In modern times invariance under the conformal group is characteristic of physical systems with no mass (e.g., a photon has no mass), and particles with no mass move at the speed of light along light rays (which are invariant under conformal transformations).

We are interested in the *conformal group* for these two geometric spaces. On M^4 we have the conformal group generated by

(a) Lorentz transformations, i.e., affine transformations that fix a specific point p and that preserve the form (2.1)
(b) translations: $x \to x + c$
(c) dilations: $x \to \alpha x$, $\alpha \in \mathbf{R}$.
(d) inversions: for a choice of coordinates

$$x \to \frac{-x}{\|x\|^2_{M^4}},$$

where $\|x\|^2_{M^4} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$. On E^4, we have the same thing but with Lorentz transformation replaced by orthogonal transformations, and the inversions involving the Euclidean norm.

The conformal transformations of these spaces have singularites in general (on a light cone or a single point $\|x\|_{M^4}$ or $\|x\|_{E^4}$ is zero in the formula for the inversions). We want to *compactify* these spaces so that the conformal groups are transformation groups of compact spaces with no singularities.

Recall that for Möbius transformations of the complex plane $\mathbf{C}$

$$z \mapsto \frac{az + b}{cz + d},\ ad - bc \neq 0,$$

we can add a point at ∞, $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, so that the Möbius transformations are nonsingular holomorphic mappings of $\bar{\mathbf{C}}$ to $\bar{\mathbf{C}}$. In this manner, the "pole" gets mapped smoothly to the point at ∞. We can do the same thing for M^4 and E^4, by adding points at ∞ corresponding to the "standard pole" for the inversion. Setting

$$M = M^4 \cup \{\text{light cone}\}$$
$$E = E^4 \cup \{\text{point}\},$$

we get compact manifolds on which the conformal groups in question act transitively and smoothly. In fact, as manifolds,

$$\mathrm{M} \cong \mathrm{U}(2) \cong S^1 \times S^3$$
$$\mathrm{M} \cong,$$
$$E \cong S^4.$$

This compactification is an old idea and has been useful in various contexts (cf. Kuiper 1949, Penrose 1965, Segal 1976, Wells 1982). Studying fields on the compactification corresponds to studying fields on M^4 or E^4 with certain specific asymptotic properties near infinity. The compactification, therefore, provides a convenient way to understand the asymptotic behavior in a systematic manner.

In classical mechanics one describes a physical system by Newton's laws ($F = ma = m\ddot{x}$). It was discovered by Hamilton that Newton's laws and differential equations describing other physical phenomena could be derived from a more fundamental quantity using the calculus of variations. In mathematics the differential equation describing a geodesic is derived by considering the curves which minimize the arc length integral. Similarly, in physics one describes a physical system by means of an *action integral,* and one minimizes this integral with respect to all possible systems (similar to minimizing over all possible curves in the case of geodesics). The integrand is called the *Lagrangian* or *Lagrangian density.* The minimum (when it exists) satisfies a system of differential equations analogous to the equations describing geodesics. In this way, for appropriate action integrals, one can derive Newton's equations, Maxwell's equations, and, as Hilbert showed, Einstein's equation of gravity. The contemporary theory of Yang-Mills equations is derived from an action principle generalizing that of classical electromagnetic theory.

Thus on Minkowski space we consider a physical system by describing a Lagrangian *density.* In classical mechanics the Lagrangian density was described as

$$L = (\text{kinetic energy}) - (\text{potential energy}),$$

where the Hamiltonian density corresponded to the total energy. In field theory, the Lagrangians are expressions in the field variables involving usually first derivatives. For example, if A_μ, $\mu = 0,1,2,3$, is the electromagnetic potential, then

$$L(A_\mu)(x) = \sum_{\mu\nu} F^{\mu\nu}(x)F_{\mu\nu}(x) = \|F(x)\|^2, \qquad \mathbf{2.4}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This is the same as

$$L(A_\mu) = |\vec{E}|^2 - |\vec{B}|^2$$

using the classical notation of $\vec{E}$ and $\vec{B}$ for the electric and magnetic fields. Here $\{F_{\mu\nu}\}$ can be thought of as the 6 independent coefficients of a 2-form

$$F = \Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu$$

in Minkowski space, which also correspond to the 6 independent coefficients of the classical vector fields $\vec{E}$ and $\vec{B}$. If ϕ denotes a generic field (scalar, vector, or spinor) on M^4, and $L(\phi)$ represents a Lagrangian density depending on ϕ, then we define

$$S(\phi) = \int_{M^4} L(\phi) d\, x$$

to be the *action* of ϕ. The *least action principle* tells us that the *classical field equations* for the physical system described by L are given by the critical points of this action in the infinite-dimensional space of all fields $\{\phi\}$, that is, those fields which satisfy the Euler-Lagrange equations for this action functional. In the example (2.4) above, the field equations are simply Maxwell's equations for an electromagnetic field.

Quantum mechanics was discovered in the 1920's by Heisenberg and Schrödinger. Over the next half century many attempts have been made to reconcile the notions of field theory (as represented by the electromagnetic field, for instance) with the quantum ideas. This circle of ideas has evolved to become what is known as *quantum field theory.*

Fundamentally, in quantum mechanics one cannot predict precisely the events of nature, only their probability of occurrence for repeated experimentation. This is true also in the context of quantum field theory. In 1949 Feynmann introduced the integral which bears his name. This is an integral over an infinite-dimensional space of configurations or possible states of a given field-theoretic system (for example, over all potential electromagnetic fields satisfying given boundary conditions). This contrasts with Hamilton's minimizing the finite-dimensional action integral $S(\phi)$ over the same set of configurations. This "averaging over configurations" yields a probabilistic quantity which is subject to experimental verification. The integral is of the form

$$\int_{\{\phi\}} e^{iS(\phi)} d\mu(\phi),$$

where $\{\phi\}$ denotes possible configurations of the system and $d\mu$ is a suitable measure on the space of such configurations. The integrand contains the structure of the physical theory, just as it does for the least action principle. This integral is still not understood in any rigorous sense; in particular, there is still no well-defined suitable measure $d\mu$. However, Feynman's scheme for "approximating the integral" yields surprisingly accurate agreement with experiment in certain theories. If we consider the corresponding *Euclidean action* $S_E(\phi)$, where we go over to E^4 by letting $x^0 \to ix^0$ as indicated above in (2.3), then the integral is of the form

$$\int_{\{\phi\}} e^{-S_E(\phi)} d\mu_E(\phi)$$

for a suitable "Euclidean measure" $d\mu_E$. This is the conversion from oscillatory to exponentially damped integral mentioned earlier. The measures $d\mu$ and $d\mu_E$ are not rigorously defined in general for 4-dimensional space. In E^2 and E^3, however, the analogous Feynman integrals have been rigorously formulated by Glimm and Jaffe in a series of important papers (see the recent book Glimm-Jaffe 1981). The corresponding 4-dimensional questions still remain open. Even though one doesn't understand the integral rigorously, the physicists have used various asymptotic "approximations" to the Feynman integral. One method involves evaluating the integrand "near" critical points, in an infinite-dimensional version of the stationary phase or saddle point approximation to an integral with an exponential integrand (over a finite-dimensional space). The critical points in this context are the solutions of the classical field equations, and using these classical solutions, one hopes to approximate the Feynman integral in a reasonable fashion.

3. The Twistor Geometry of Penrose

Twistor geometry is a tool which provides a relationship of the following sort:

$$\left\{\begin{array}{l}\text{several complex} \\ \quad\text{variables} \\ \text{algebraic geometry} \\ \text{algebraic topology}\end{array}\right\} \underset{\text{geometry}}{\overset{\text{twistor}}{\longleftrightarrow}} \left\{\begin{array}{l}\text{solutions of} \\ \text{classical} \\ \text{field equations}\end{array}\right\} \qquad 3.1$$

It provides a mechanism for transcribing problems concerning classical field equations (solutions, existence, uniqueness, asymptotic behavior, boundary value problems, etc.) into corresponding problems in the interrelated areas of algebraic geometry, several complex variables, and algebraic topology. This is similar to the Fourier transform which converts linear problems involving differentiation (partial differential equations) into corresponding algebraic problems, some of which are soluble, giving information (via an inverse transform) about the original problems.

Let's briefly discuss the nature of the mathematics involved in the left-hand side of (3.1). The oldest discipline is algebraic geometry, which is historically related to solving algebriac equations of various degrees; one studies the zeros of polynomials which define algebraic varieties. These may have singularities or not, depending on the Jacobian of the defining polynomials. There is the local behavior of such varieties, which relates to commutative ring theory, and the global behavior which relates to the global topological behavior as well as to moduli problems. Moduli problems refer to variable algebraic (or analytic) structures on the same underlying topological space; this goes back to questions first studied by Riemann.

Algebraic topology is concerned with global invariants of topological spaces. Here one attempts to classify topological manifolds, for instance, by a (usually countable) set of numbers (called invariants) which depend only on the topological type and not on the particular presentation of a given manifold or space (for example, the number of handles in a closed 2-dimensional surface is such an invariant).

In several complex variables we have a generalization of the "geometry" in algebraic geometry in that we study the zeros of holomorphic functions which have locally convergent power series expansions in terms of local complex coordinates. Thus polynomials go over to power series, and we have transcendental phenomena to contend with. While the analytic theory is vastly different, the geometry is remarkably similar to that of algebraic geometry. This is due to a remarkable theorem of Weierstrass which asserts that locally a holomorphic function looks like a polynomial in one variable with coefficients in the remaining variables multiplied by a nonvanishing function. So the zeros of holomorphic functions have polynomial-like behavior in one variable (locally), and this greatly facilitates their study.

In this area there is a chain of abstractions which has the following hierarchy:

a) holomorphic functions in **C**.
b) holomorphic functions in $\mathbf{C}^n$.
c) vector-valued holomorphic functions in $\mathbf{C}^n$.
d) holomorphic and meromorphic functions on complex manifolds (locally as in a), b) and c)).
e) sections of vector bundles on complex manifolds; these are global objects on a manifold X which are locally like c), but the vector-valued function representation can vary from one local open set to another, and there are transition functions which are part of the data of a vector bundle.

This latter concept of (holomorphic) vector bundles on (complex) manifolds is of recent origin and plays a significant role in the formulation of fundamental results like the Hirzebruch-Riemann-Roch theorem (see, e.g., Hirzebruch 1966) or the more general Atiyah-

Singer Index Theorem (Atiyah-Singer 1963), which is a relationship between suitable measures of the number of global solutions of a given differential equation on a manifold X and topological invariants of the vector bundles and the manifold X itself. There is a strong interplay between analysis (local differential calculus, integration, and differential equations) and topology in all this.

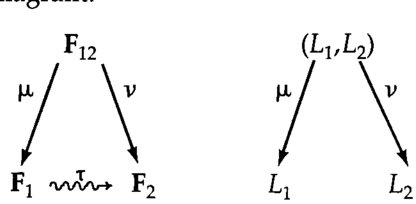
Out of the study of vector bundles came cohomology theory, and later sheaf theory. These were used to study the relationship between local and global solutions to specific geometric problems. It is a systematic language for studying local and global function-theoretic and geometric data and their interaction. In the period 1945–1965 there were great developments in all of these areas, capping several centuries of research on the fundamental phenomena arising from the study of polynomials, global geometric spaces, and complex-analytic phenomena. This is the language into which the twistor geometry of Penrose translates many problems of mathematical physics. But just as in the case of the Fourier transform, translating a specific problem into a new language doesn't always make the problem more tractable. When it does, however, it can change the perspective on a given problem by a vast amount.

Now that we've discussed the two areas for which twistor geometry provides a connection we can briefly describe it. We start out with a four-dimensional complex vector space $\mathbf{T}$ equipped with an Hermitian form ϕ of signature $(+ + - -)$; that is, in suitable coordinates (Z^0,Z^1,Z^2,Z^3) in $\mathbf{T}$, $\phi(Z) = |Z^0|^2 + |Z^1|^2 - |Z^2|^2 - |Z^3|^2$. We call $\mathbf{T}$ the space of *twistors*, or *twistor space*. From the geometric data of $\mathbf{T}$ we can describe various other spaces which will in turn be related to each other.

Let L_j denote a complex subspace of $\mathbf{T}$ of dimension j. We will consider certain "spaces of subspaces" defined as follows:

$\mathbf{F}_1 := \{L_1 \subset \mathbf{T}\}$ (all 1-dimensional subspaces)
$\mathbf{F}_2 := \{L_2 \subset \mathbf{T}\}$ (all 2-dimensional subspaces)
$\mathbf{F}_{12} := \{(L_1,L_2): L_1 \subset L_2 \subset \mathbf{T}\}$ (all nested subspaces of these dimensions).

We see that $\mathbf{F}_1 \cong \mathbf{P}_3(\mathbf{C})$, 3-dimensional complex projective space, while $\mathbf{F}_2$ is a Grassmannian manifold and $\mathbf{F}_{12}$ is a type of manifold called a flag manifold. These spaces are all compact complex manifolds (they all are generalizations of projective space, which is well known to be compact). They are related by the following diagram:



3.2

where the mappings μ and ν are described by forgetting the second or first component of a given point in $\mathbf{F}_{12}$ (respectively). One can check that $\dim_{\mathbf{C}}\mathbf{F}_{12} = 5$, $\dim_{\mathbf{C}}\mathbf{F}_1 = 3$, $\dim_{\mathbf{C}}\mathbf{F}_2 = 4$, and that the fibres of μ are isomorphic to $\mathbf{P}_2(\mathbf{C})$ while the fibres of ν are isomorphic to $\mathbf{P}_1(\mathbf{C})$. This *double fibration* (3.2) provides the link between Minkowski space and 3-dimensional complex projective space (the space $\mathbf{F}_1$), as we will see below.

First we note that the double fibration provides us with a *correspondence* τ between the spaces $\mathbf{F}_1$ and $\mathbf{F}_2$, which is a generalization of a mapping. Namely, to each point $x \in \mathbf{F}_1$ one can associate $\tau(x)$, a *subject* of $\mathbf{F}_2$, and conversely, for each point $y \in \mathbf{F}_2$, there is a subset $\tau^{-1}(x)$ of $\mathbf{F}_1$. These are defined by

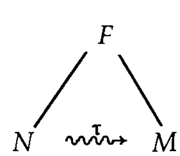
$$\tau(x) := \nu \circ \mu^{-1}(x) \cong \mathbf{P}_2(\mathbf{C}),$$
$$\tau^{-1}(x) := \mu \circ \nu^{-1}(x) \cong \mathbf{P}_1(\mathbf{C}).$$

The nature of the geometry is that the inverse image of a point under one of the two mappings μ or ν is isomorphic to a projective space, and the remaining mapping pushes the projective space down to the opposite space in an isomorphic fashion (here isomorphic refers to complex-analytic equivalence).

We now use the Hermitian form ϕ, part of the twistor structure, to complete the picture. Namely, if L_j is a subspace of $\mathbf{T}$ we define $\phi(L_j) = 0$ if $\phi(v) = 0$ for all $v \in L_j$. Then we define

$$M = \{L_2 \in \mathbf{F}_2: \phi(L_2) = 0\}$$
$$N = \{L_1 \in \mathbf{F}_1: \phi(L_1) = 0\}$$
$$F = \{(L_1,L_2) \in \mathbf{F}_{12}: \phi(L_2) = 0\}$$

Now $GL(\mathbf{T})$ acts on $\mathbf{T}$ and induces an action on subspaces and hence on the flag manifolds $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_{12}$. If we let G be the unimodular subgroup of $GL(\mathbf{T})$ which preserves the form ϕ ($G \cong SU(2,2)$), then G acts on the flag manifolds also, and preserves the zero sets of ϕ (M, N, and F). In fact the action on M, N, and F is transitive, and these are smooth real submanifolds of the corresponding complex manifolds. The submanifold M with the group action G can be identified with compactified Minkowski space with the action of the conformal groups (see, for example, Wells 1982). Thus we see Minkowski space arising from the twistor structure. Moreover, $N \cong S^2 \times S^3$ and can be identified with the space of light rays in compactified Minkowski space, and the action of G is the action of the conformal group on the space of light rays. The correspondence τ restricted to these zero sets yields



$\tau(x)$ = light ray in M ($\cong S^1$)
$\tau^{-1}(x)$ = celestial sphere
= all light rays passing through y
$\cong \mathbf{P}_1(\mathbf{C})$

As mentioned before, Penrose's main idea is to replace M, Minkowski space (now compactified) with N, the space of light rays. But moreover, $M \subset \mathbf{F}_2$, a 4-dimensional complex manifold, which locally is a complexification of M, and one can show that the 4-sphere S^4 of Euclidean conformal geometry is also embedded naturally in $\mathbf{F}_2$ (see Wells 1982) so that $\mathbf{F}_2$ is also a complexification of S^4. Thus the concept of analytic continuation of fields from E^4 to M^4 can take place in an affine piece of $\mathbf{F}_2$. We call $\mathbf{F}_2$ *complexified compactified Minkowski space* and denote it by $\mathbf{M}$. This complexification is important in the concept of positive frequency and quantum field theory (see for example, Streater-Wightman 1978), and is also a natural part of the twistor geometry. The space N of *null projective twistors* (identified with the set of all light rays in M) is naturally a real 5-dimensional hypersurface in $\mathbf{F}_1(\cong \mathbf{P}_3(\mathbf{C}))$, and this gives N a *tangential Cauchy-Riemann structure* (see, e.g., Wells 1974). This is a reflection of the twistor geometry and plays a significant role in understanding the geometric objects on N (or defined near N in $\mathbf{F}_1$) which corresponds to quantities of physical interest on M.

If we consider solutions of differential equations on M which are real-analytic, then they will be complex-analytic on some region in the complexification $\mathbf{M}$. We want to indicate schematically how data on $\mathbf{F}_1$ can be transformed into data on $\mathbf{F}_2$ and how on $\mathbf{F}_2$ this must satisfy some differential equation. We will consider a linear situation because it's simpler, but the basic idea generalizes to nonlinear phenomena as well. Namely, if f is a differential form on $\mathbf{F}_1$ (of an appropriate type, which will not be spelled out here, see Wells 1982), then the pullback μ^*f will be a differential form on $\mathbf{F}_{12}$, and if d_μ denotes *exterior differentiation along the fibres of* μ, then since μ^*f is *constant along the fibres*, we see that

$$d_\mu(\mu^*f) = 0. \tag{3.3}$$

Now perform a fibre integration with respect to the second fibration in our double fibration, defining

$$\phi(x) = \int_{\nu^{-1}(x)} \mu^*f, \; x \in \mathbf{F}_2, \tag{3.4}$$

then ϕ, suitably interpreted, is (at least locally) a vector-valued function on $\mathbf{F}_2(=\mathbf{M})$. Explicitly, these integrals (3.4) have the following form. If $(x^0,x^1,x^2,x^3) = x^a$ are coordinates for Minkowski space, then let

$$(x^a,\zeta),\; \zeta \in \mathbf{C}$$

be local coordinates for F and let (Z^0,Z^1,Z^2,Z^3) be homogeneous complex coordinates for $\mathbf{F}_1 \supset N$. The mapping μ in (3.2) is given by

$$\begin{aligned} Z^0 &= (x^0+x^1)\zeta + (x^2 + ix^3) \\ Z^1 &= (x^2-ix^3)\zeta + (x^0-x^1) \\ Z^2 &= \zeta \\ Z^3 &= 1. \end{aligned}$$

We now consider the function

$$f(Z) = \frac{1}{Z^0Z^1}$$

and form the integral

$$\begin{aligned} \phi(x) &= \frac{1}{2\pi i}\oint f(Z(x,\zeta))d\zeta \\ &= \frac{1}{2\pi i}\oint \frac{d\zeta}{Z^0(x,\zeta)Z^1(x,\zeta)}, \end{aligned}$$

where the contour integral is taken around the poles in the integrand. By calculating the residue at one of the two poles we find that

$$\phi(x) = \frac{1}{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}.$$

Then $\phi(x)$ will satisfy the wave equation

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2\right]\phi(x) = 0,$$

and this is also true for *any* meromorphic function of the same type homogeneous of degree -2. Moreover, *all* solutions of the wave equation can be generated in this fashion. If we consider homogeneous functions of degree -4, we can generate vector-valued functions in a similar manner which will generate solutions to Maxwell's equations. As indicated earlier, all solutions to Maxwell's equations arise this way. To prove this, it's necessary to reformulate the contour integrals in a more systematic manner using an appropriate cohomology theory, as two distinct integrands can yield the same solution of the field equations, and we need appropriate equivalence relations (cohomology).

In the geometry of this double fibration the differentiation operator d_μ is "transversal" to the fibre-integration operator $\int_{\nu^{-1}(x)}$, and it follows that there is an operator ∇ induced on the vector-valued functions satisfying

$$\nabla \left(\int_{\nu^{-1}(x)} g\right) = \int_{\nu^{-1}(x)} d_\mu g,\; x \in \mathbf{F}_2,$$

that is, we can invert the order of differentiation and integration for a suitable operator ∇. But then it follows from (3.3) and (3.4) that

$$\nabla\phi(x) \equiv 0,$$

and so, ϕ defined by (3.4) automatically satisfies a differential equation induced from the geometry (namely, ∇ is induced from d_μ, and the fibration ν and d_μ is induced from the fibration μ). For instance, it can be shown that for appropriate classes of differential forms on $\mathbf{F}_1$, the induced operator ∇ gives Maxwell's equations, and the vector-valued functions ϕ defined by (3.4) generate all solutions of Maxwell's equations by such an integral formula, where the data f is essentially free and is not required to satisfy any differential equation. In fact, f depends on 3 complex parameters (coordinates on $\mathbf{F}_1$), just as the Cauchy-hypersurface hyperbolic theory would predict.

In recent years there has been much progress in using the Penrose transform to generate solutions of various classical field theories of interest in contemporary physics. Without any details, we mention only that most of the transforms involve nonlinear data. The nature of the double fibration (3.2) allows one to "pull back" data for μ and "push down" data by ν, even if it is nonlinear in nature. Fundamentally, one transforms vector bundles to vector bundles, and the transition functions of the transformed bundle involve matrix-valued analogues of the contour integral formulae written down above.

These ideas have been applied to various contexts. First there are the linear field theories, where Penrose's ideas first came to light (see Penrose-MacCallum 1972). This includes the field theories which describe photons (Maxwell's equations) and neutrinos (Dirac 1928, Weyl 1929). More generally, there are the nonlinear theories including Yang-Mills fields in the study of elementary particles (Yang-Mills 1954), Einstein's gravitational fields, and more recently the fields which describe monopoles (see Jaffe-Taubes 1980). A recent lecture note volume (Wells 1982) gives a detailed survey of the interaction between complex geometry, classical field theory, and the Penrose transform. An earlier volume (Atiyah 1979) gives a more detailed look at the special case of the elegant solution (due to Atiyah, Hitchin, Drinfeld and Manin) to the instanton problem in the context of Yang-Mills fields by means of twistor geometry.

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methods devised to attack it have been used to prove a wealth of interesting results. I do not know how the theory will be in future. It is quite conceivable that deBranges' work has given it new interest and that it will grow in new directions.

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> Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity one cannot understand either the aims or the achievements of mathematics in the last fifty years.
>
> H. Weyl, in 1950