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The Inverse Penrose Transform of a Solution to the Maxwell–Dirac–Weyl Field Equations

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An explicit family of solutions to the nonlinear coupled Maxwell–Dirac–Weyl equations in Minkowski space is presented. The abstract results of Henkin and Manin (*Phys. Lett. B* **95** (1980), 405–408) show that these solutions are equivalent by the Penrose transform to a coupled system of cohomology classes and a complex line bundle on ambitwistor space, the space of null lines in Minkowski space. The explicit inverse Penrose transform of this family of solutions is computed giving explicit expressions for the line bundle (transform of the vector potential), the obstruction to extension (transform of the charge), and the two cohomology classes (transform of the Dirac–Weyl coupled spinor fields).

1. INTRODUCTION

Recently Henkin and Manin [8] showed how to transform solutions of coupled Yang–Mills–Dirac–Higgs systems arising in contemporary gauge models of elementary particles to solutions of coupled algebraic equations involving vector bundles, extensions of vector bundles, and cohomology classes defined on a quadric hypersurface in $\mathbf{P}_3(\mathbf{C}) \times \mathbf{P}_3(\mathbf{C})$. This result is one example of the general Penrose transform mapping problems involving algebraic–geometric and complex-analytic data to solutions of classical field equations on Minkowski or Euclidean space (see Wells [18], Atiyah [1] for surveys of recent work in this area). The paper of Henkin and Manin is of an

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abstract nature, and the purpose of this paper is to present a detailed example of the Penrose transform in the context of coupled nonlinear field equations. Previous examples in the literature deal with, for instance, linear equations (Penrose & MacCallum [12]), Einstein's self-dual equations (e.g., Ward [14]), or the self-dual Yang–Mills equations (instantons, see Atiyah [1]), and there are others. All of them do not directly involve nontrivial source terms in the Yang–Mills equations. In this paper we consider the coupled Maxwell–Dirac–Weyl equations on complexified Minkowski space and ignore reality questions, as we are interested primarily at this point in the Penrose transform of the complex-analytic data. As in all other such Penrose transforms (e.g., the ADHM method for contructing instantons, see Atiyah [1]), the reality conditions can be realized as an additional constraint on the problem after the complex-analytic problem has been resolved. Our purpose here is to attempt to understand in twistor-geometric terms the nature of the nonlinear coupling process. This example will perhaps be a prototype of more general behavior.

In Section 2 we present the example in Minkowski space, and discuss some of its properties. It has a character similar to plane wave solutions of the nonlinear Einstein equations, and hence we call it a "plane-wave" solution of the coupled system. The solution consists of a Maxwell potential $V_{AA'}$ and a pair of spinor fields ϕ_A and $\psi_{A'}$ of an appropriate type. The solution has a distinct null vector associated with it, and the "wave property" is transversal to this propogation direction. In Section 3 we summarize briefly the twistor geometry and the Penrose transform formalism necessary for our construction. In Section 4 we construct explicitly the inverse Penrose transform of the plane-wave solution, which becomes a triple (L, Φ, Ψ) consisting of a holomorphic line bundle, and two cohomology classes and such that

$$\mathscr{P}(L) = V_{AA'}, \qquad \mathscr{P}(\Phi) = \phi_A, \qquad \mathscr{P}(\Psi) = \psi_{A'},$$

where $\mathscr{P}$ denotes the generic Penrose transform. The coupling of the spinor fields ϕ_A, $\psi_{A'}$, with the Maxwell potential $V_{AA'}$ is the Penrose transform of the coupling of (L, Φ, Ψ). We also investigate the dependence of the coupling on a coupling constant and obtain asymptotic expansions of some of the geometric objects involved in terms of solutions of associated linear problems. It would be interesting to relate this family of examples to Henkin's scheme [7] for transforming such coupled problems involving bundles and cohomology to problems involving simple bundles on more general spaces.

2. A Plane-Wave Solution of the Maxwell–Dirac–Weyl Field Equations

The field equations we want to study are the coupled Maxwell–Dirac–Weyl equations

$$\begin{aligned} (\nabla^{AA'} - \lambda V^{AA'})\phi_A &= 0 \\ (\nabla^{AA'} + \lambda V^{AA'})\psi_{A'} &= 0 \\ d^*F &= \lambda \phi_A \psi_{A'}\, dz^{AA'}, \end{aligned} \tag{2.1}$$

where

$$V = V_{AA'}\, dz^{AA'}$$

is the Maxwell potential, $F = dV$ is the Maxwell field strength, and λ is a coupling constant. Here we denote by $z^{AA'}$ spinor coordinates for complexified Minkowski space–time (cf., e.g., Penrose & MacCallum [12]), which we denote by $\mathbb{M}^I$ ($\cong \mathbf{C}^4$). We will denote by $\mathbb{M}$ compactified complexified space time and the superscript I refers to the deletion of the points at infinity (see Sect. 3). We denote by $\nabla_{AA'} = \partial/\partial z^{AA'}$ the differentiation with respect to the coordinates above. Real affine Minkowski space M^4 corresponds to $z^{AA'}$ being Hermitian ($\cong \mathbf{R}^4$). The field equations (2.1) arise from the Lagrangian

$$\mathscr{A} = \int_{M^4} \{|F|^2 + [\psi_{A'}(\nabla^{AA'} - \lambda V^{AA'})\phi_A - \phi_A(\nabla^{AA'} + \lambda V^{AA'})\psi_{A'}]\}\, d^4x \tag{2.2}$$

which is a $GL(1, \mathbf{C}) = \mathbf{C}^*$ gauge theory. We are considering holomorphic fields on $\mathbb{M}^I$ and ignoring reality and unitary conditions on the gauge theory at this time.

We have a family of solutions of (2.1) which goes as follows (letting $\lambda = 1$ for the time being). Consider a fixed null direction in M^I, say $\nu^{AA'}$, then $\nu^{AA'}$ is the product of constant spinors, $\nu^{AA'} = \phi^A \psi^{A'}$. Consider potentials $V_{AA'}$ of the form

$$V_{AA'}(z) = \phi_A \psi_{A'} U(z),$$

where $U(z)$ is a scalar field. Then choose coordinates such that

$$z^{AA'} = \begin{pmatrix} u & \zeta \\ \tilde{\zeta} & v \end{pmatrix}$$

so that u is the coordinate in the $\nu^{AA'}$ direction.

LEMMA 2.1. *If* $U(z) = U(u, v, \zeta, \tilde{\zeta})$ *satisfies*

(a) $\partial U/\partial u = 0$,

(b) $\partial^2 U/\partial\zeta\,\partial\tilde{\zeta} = 1$,

then the triple

$$\phi_A, \psi_{A'}, V_{AA'} = \phi_A \psi_{A'} U(z) \tag{2.3}$$

satisfies the field equations (2.1).

The proof of this lemma is an elementary calculation which is omitted.

Remark. The equations (2.1) are conformally invariant solutions of the form in (2.3) can be composed with conformal transformations to generate a large family of solutions. The constant spinors ϕ_A and $\psi_{A'}$ then become elementary states.

We want to consider a special case of (2.3), where $U(z)$ is a simple algebraic expression, as we want to consider in detail its inverse Penrose transform. Namely, let us consider, in any coordinate system, a null vector $v = \phi_A \psi_{A'}$, where ϕ_A and $\psi_{A'}$ are constant spinors, as before. Choose a vector $n = n^a = n^{AA'}$ satisfying

$$n \cdot n = 1, \qquad n \cdot v = 0,$$

where $\xi \cdot \eta = \xi^a \eta_a = \xi^{AA'}\eta_{AA'}$ is the usual bilinear pairing of vectors in complex Minkowski space (the **C**-linear extension of the Minkowski inner product on M^4). Then we see that

$$\{\phi_A, \psi_{A'}, V_{AA'} = \phi_A \psi_{A'} (n \cdot z)^2\} \tag{2.4}$$

is a solution to (2.1).

3. THE PENROSE TRANSFORM ON AMBITWISTOR SPACE

Twistor space $\mathbb{T}$ is, by definition, a 4-dimensional complex vector space equipped with an Hermitian form Φ of signature $(++--)$. From the space $\mathbb{T}$ one constructs various compact complex manifolds with reality conditions induced from Φ, and relations between them. In our discussion of holomorphic fields in this paper we will ignore the reality conditions, but note that one can equip $\mathbb{T}$ with various real stuctures leading to a variety of reality conditions on the derived complex manifold (cf., Wells [18]).

Given twistor space $\mathbb{T}$ we construct various flag manifolds associated with it. Let

$$\mathbb{F}_{i_1 i_2, \ldots, i_d} = \{L_{i_1}, \ldots, L_{i_d} : L_{i_k} \text{ are subspaces of } \mathbb{T} \text{ of dimension } i_k \\ \text{and } L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_d}\}.$$

This is a flag manifold of type $(i_1,\dots,i_d)$ (cf., Wells [17]). In general if

$$\{i_1,\dots,i_d\} = \{j_1,\dots,j_k\} \cup \{l_1,\dots,l_{d-k}\},$$

is a disjoint union of subsets of $\{1,\dots,4\}$, then we have a double fibration

$$\begin{array}{ccc} & \mathbb{F}_{i_1,\dots,i_d} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbb{F}_{j_1,\dots,j_k} & & \mathbb{F}_{l_1,\dots,l_{d-k}} \end{array}$$

where μ and ν are the natural projections mappings. For instance, one has

$$\begin{array}{ccc} & \mathbb{F}_{12} & \\ \swarrow & & \searrow \\ \mathbb{F}_1 & & \mathbb{F}_2 \end{array} \qquad \begin{array}{ccc} & \mathbb{F}_{23} & \\ \swarrow & & \searrow \\ \mathbb{F}_3 & & \mathbb{F}_2 \end{array}$$

and $\mathbb{F}_2$ is identifiable with complexified compactified Minkowski space (Wells [16, 18]) and $\mathbb{F}_1 = \mathbf{P}(\mathbb{T}) \cong \mathbf{P}_3(\mathbf{C})$ is projective twistor space, and $\mathbb{F}_3 = \mathbf{P}(\mathbb{T}^*) \cong \mathbf{P}_3(\mathbf{C})$ is dual projective space. This is the famous Klein correspondence, and it has been used in the representation of massless fields (Eastwood *et al.* [3]), in the description of self-dual Einstein's equations (Penrose [11]), and for monopoles (Ward [15]). A natural generalization of these two double fibrations is given by

$$\begin{array}{ccc} & \mathbb{F}_{123} & \\ {}^{\sigma}\swarrow & & \searrow^{\rho} \\ \mathbb{F}_{13} & & \mathbb{F}_2 \end{array}$$

and we see that $\mathbb{F}_{13}$ is naturally embedded in $\mathbb{F}_1 \times \mathbb{F}_3$ by the inclusion

$$(L_1, L_3) \to (L_1, L_3),$$

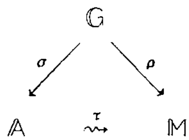
and $\mathbb{F}_{13}$ is a quadric surface. In terms of homogenous coordinates $\{Z^\alpha\}$ for $\mathbb{F}_1$ and $\{W_\beta\}$ for $\mathbb{F}_{3'}$, where $Z \cdot W = Z^\alpha W_\alpha$ is the natural duality pairing of $\mathbb{T}$ and $\mathbb{T}^*$, we see that

$$\mathbb{F}_{13} = \{[Z^\alpha], [W_\beta] \in \mathbb{F}_1 \times \mathbb{F}_3 : Z \cdot W = 0\}.$$

Let us denote:

$\mathbb{A} := \mathbb{F}_{13}$ = ambitwistor space,
$\mathbb{M} := \mathbb{F}_2$ = Minkowski space,
$\mathbb{G} := \mathbb{F}_{123}$ = correspondence space for $\mathbb{A}$ and $\mathbb{M}$.

Thus we have



and τ is the correspondence between $\mathbb{A}$ and $\mathbb{M}$, $\tau(x)=\rho\circ\sigma^{-1}(x)$ is a null line in $\mathbb{M}$ with respect to the natural conformal metric on $\mathbb{M}$, and $\tau^{-1}(y)=\sigma\circ\rho^{-1}(y)\cong\mathbf{P}_1\times\mathbf{P}_1$ is a complex quadric embedded in the ambitwistor space $\mathbb{A}$. The correspondence τ is the *twistor correspondence* in this setting. Ambitwistor space then parametrizes the null lines in $\mathbb{M}$, and is often called the *linespace* of the space of lines, denoted by $L(\mathbb{M})$ (cf. e.g., Henkin & Manin [8]).

We want to work on affine Minkowski space, and on the corresponding regions in $\mathbb{A}$ and $\mathbb{G}$. So let I be a specific 2-dimensional subspace of $\mathbb{T}$, and let

$$\mathbb{M}^I=\{L_2 : L_2\cap I\neq 0\},$$

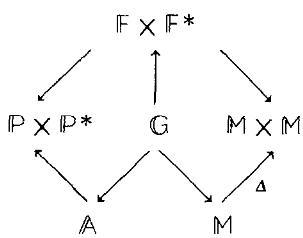
then $\mathbb{M}^I\cong\mathbf{C}^4$ and is a coordinate chart for $\boldsymbol{M}$ [6]. Let $\mathbb{G}^I=\rho^{-1}(\mathbb{M}^I)$ and $\mathbb{A}^I=\sigma(\mathbb{G}^I)$. Moreover, we have a natural embedding of

$$\mathbb{G}=\mathbb{F}_{123}\subset\mathbb{F}_{12}\times\mathbb{F}_{23},$$

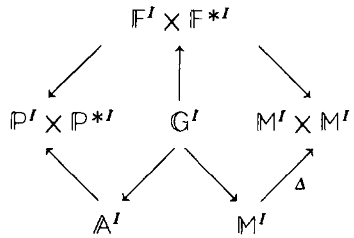
and if we denote

$$\mathbb{F} := \mathbb{F}_{12},\qquad \mathbb{F}^* := \mathbb{F}_{13},$$
$$\mathbb{P} := \mathbb{F}_1,\qquad \mathbb{P}^* := \mathbb{F}_3,$$

we have the following geometric diagram



where Δ is the diagonal mapping. We have a similar diagram when we restrict our attention to affine Minkowski space of the form



We will introduce suitable coordinate systems on all of these spaces as we need them.

The embedding $\mathbb{A} \to \mathbb{P} \times \mathbb{P}^*$ induces a sequence of formal neighborhoods of $\mathbb{A}$ in $\mathbb{P} \times \mathbb{P}^*$ of the following form. Let $\mathscr{I}_{\mathbb{A}}$ be the ideal sheaf in $\mathscr{O}_{\mathbb{P} \times \mathbb{P}^*}$ of local holomorphic functions on $\mathbb{P} \times \mathbb{P}^*$ which vanish on $\mathbb{A}$. Then

$$\mathscr{O}_{\mathbb{A}} = \mathscr{O}_{\mathbb{P} \times \mathbb{P}^*} / \mathscr{I}_{\mathbb{A}}$$

is the structure sheaf of $\mathbb{A}$ (local holomorphic functions on $\mathbb{A}$), and we define

$$\mathscr{O}_{\mathbb{A}}^{(k)} := \mathscr{O}_{\mathbb{P} \times \mathbb{P}^*} / \mathscr{I}_{\mathbb{A}}^{k+1}$$

to be the structure sheaf of the kth formal neighborhood of the embedding $\mathbb{A} \subset \mathbb{P} \times \mathbb{P}^*$ (cf., Griffiths [4, 5]). We let $\mathbb{A}^{(k)} = (\mathbb{A}, \mathscr{O}_{\mathbb{A}}^{(k)})$. This corresponds locally to polynomials of degree k in the normal coordinates of the embedding with holomorphic coefficients in the local coordinates of $\mathbb{A}$. Then a formal neighborhood $\mathbb{A}^{(k)}$ is, as a point set, the same as $\mathbb{A}$, but the structure sheaf of functions on $\mathbb{A}^{(k)}$ reflects polynomial information in the normal direction up to order k. There is a natural restriction mapping

$$(U, \mathscr{O}_{\mathbb{P} \times \mathbb{P}^*}) \to (\mathbb{A}, \mathscr{O}_{\mathbb{A}^{(k)}}),$$

where U is any topological neighborhood of $\mathbb{A}$ in $\mathbb{P} \times \mathbb{P}^*$ (cf, Griffiths [4, 5]).

If we have any geometric or analytic object f on $\mathbb{A}$ (or $\mathbb{A}^I$), e.g., a vector bundle, a sheaf, a function, or a cohomology class, then we can consider the *extension problem* of extending f to a topological neighborhood U of $\mathbb{A}$, or to all of $\mathbb{P} \times \mathbb{P}^*$. The method of formal power series allows one to describe such extensions order by order by considering extensions first to $\mathbb{A}^{(1)}$, then to $\mathbb{A}^{(2)}$, etc. If one cannot extend to $\mathbb{A}^{(k)}$ for some k, then one can certainly not extend to a topological neighborhood U. This is described in detail by Griffiths [4, 5] for various geometric situations, and has been applied to the study of Yang–Mills fields in the work of Isenberg–Yasskin–Green [9], Witten [19], Henkin and Manin [8], and Pool [13]. The definitive result (first formulated in Henkin and Manin [8] is that there is a one-to-one correspondence between: (1) Yang–Mills fields given by a Yang–Mills potential V on a vector bundle $E_V \to \mathbb{M}^I$ and (2) vector bundles E over $\mathbb{A}^{I(2)}$, and the Yang–Mills current $J = D_V^* F$, where D_V is the covariant derivative associated with V, corresponds to the obstruction $\omega(E)$ of extending E to $A^{I(3)}$, under a certain canonical group isomorphism. The obstruction $\omega(E)$ is an element of $H^2(\mathbb{A}^I, \operatorname{Hom}(E, E) \otimes (N^*)^3)$, where N^* is the conormal bundle (whose sheaf of holomorphic sections is given by $\mathcal{I}_{\mathbb{A}}/\mathcal{I}_{\mathbb{A}}^2$), $(N^*)^p$ denotes the pth symmetric tensor product of N^* with itself, and $\operatorname{Hom}(E, E)$ denotes the bundle of homomorphisms of the vector bundle E. This is described in Henkin and Manin [8], Manin [10], and Pool [13].

The mapping

$$E \to E_V = \mathscr{P}(E)$$

is the Penrose transform of the bundle E on $\mathbb{A}^I$ to the Yang–Mills bundle E_V on $\mathbb{M}^I$. Abstractly,

$$E_V = \rho_*^0 \sigma^* E,$$

where we consider E and E_V as locally free sheaves, and ρ_*^0 is the 0th direct image sheaf. The covariant derivative operator is induced from the exterior differential operator d_σ acting on the relative deRham complex

$$\Omega^0_{\mathbb{G}/\mathbb{A}} \xrightarrow{d_\sigma} \Omega^1_{\mathbb{G}/\mathbb{A}} \to 0$$

which is extended to having coefficients in $\sigma^* E$

$$\Omega^0_{\mathbb{G}/\mathbb{A}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(\sigma^* E) \xrightarrow{d_\sigma} \Omega^1_{\mathbb{G}/\mathbb{A}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(\sigma^* E) \to 0$$

which is well defined since d_σ annihalates the transistion functions of $\sigma^* E$ on $\mathbb{G}$.

In Eastwood *et al.* [3] it was shown how cohomology classes on $\mathbb{P}^I$ map to solutions of the Dirac–Weyl operator on $\mathbb{M}^I$. This was generalized to the ambitwistor setting (see Eastwood [3], Henkin & Manin [8], Pool [13]), and our object in the remainder of the paper is to calculate explicitly the inverse Penrose transform of the 4-tuple

$$\phi_A, \psi_{A'}, V_{AA'} = \phi_A \psi_{A'} (n \cdot z)^2 \qquad \text{on (the trivial bundle on) } \mathbb{M}^I,$$
$$J = \lambda \phi_A \psi_{A'} \, dz^{AA'}. \tag{3.1}$$

In general, Eqs. (3.1) can be interpreted over an ambitwistor space as follows. There are three parts to the amibitwistor description comprising (a) the potential V, (b) the minimially coupled Dirac equations for ϕ_A and $\psi_{A'}$, and (c) the equation for the current $d^* F$.

Part (a). The Penrose transform gives

$$\mathscr{P} : H^1(\mathbb{A}^I, \mathcal{O}) \to \Gamma(\mathbb{M}^I, \Omega_1)/d(\Gamma(\mathbb{M}^I, \Omega^0)).$$

The space on the right-hand side can be regarded as the possible connections on the trivial bundle, a 1-form being thought of as a connection 1-form with the freedom to modify by an exact 1-form being precisely the gauge freedom incurred by choosing a different trivialization. This is also how the Maxwell potential V should be regarded. The image of the corresponding element of $H^1(\mathbb{A}^I, \mathcal{O})$ under

$$\exp : H^1(\mathbb{A}^I, \mathcal{O}) \to H^1(\mathbb{A}^I, \mathcal{O}^*)$$

will be denoted L. It is a line bundle on $\mathbb{A}^I$ (the generalized "twisted photon"), specializing the case of the vector bundle E on $\mathbb{A}^I$ which describes the Yang–Mills bundle E_V as indicated above.

Part (b). Consider, for example, the equation $(\nabla^{AA'} + V^{AA'})\psi_{A'} = 0$. To be a little more precise about the meaning of this equation, let $\mathscr{S}_-$ denote the sheaf of sections of the universal or tautological bundle over $\mathbb{M}$ (i.e., for each $p \in \mathbb{M}$ this bundle assigns the vector subspace $S \subset \mathbb{T}$ which defines p). If we let $\mathscr{S}_+ = (\mathcal{O}_{\mathbb{M}} \otimes \mathbb{T}/\mathscr{S}_-)^*$ then

$$\Omega^1_{\mathbb{M}} = \mathscr{S}_+ \otimes \mathscr{S}_-$$

(in the notation of Eastwood *et al.* [3], this equation would be written $\mathcal{O}_a = \mathcal{O}_{A'} \otimes \mathcal{O}_A$). The conformally invariant Dirac operator $\nabla^{AA'}$ is, strictly speaking, a differential operator (letting $\mathscr{S}^+ = (\mathscr{S}_+)^*$),

$$\mathscr{D} : \mathscr{S}_- \otimes \textstyle\bigwedge^2 \mathscr{S}_- \to \mathscr{S}_+ \otimes \textstyle\bigwedge^2 \mathscr{S}_- \otimes \textstyle\bigwedge^2 \mathscr{S}_- \cong \mathscr{S}^+ \otimes \textstyle\bigwedge^2 \mathscr{S}_-.$$

The line bundle $\bigwedge^2 \mathscr{S}_-$ is called a conformal weight (for details see Eastwood *et al.* [3]). Over $\mathbb{M}^I$ it can be, and usually is, trivialized—this is automatic if $\mathbb{M}^I$ is endowed with a metric rather than only a conformal metric. Thus, we shall more naively take $\mathscr{D}$ as an operator

$$\nabla^{AA'} : \mathscr{S}_- \to \mathscr{S}^+$$

and bear in mind that if the appropriate conformal weights are included then the Eqs. (3.1) are conformally invariant. The coupled Dirac operator $\nabla^{AA'} + V^{AA'}$ (formed by tensoring with the connection on the trivial bundle given by V) will be denoted by

$$\mathscr{D}_V : \mathscr{S}_- \to \mathscr{S}^+.$$

The coupled equation $(\nabla^{AA'} + V^{AA'})\psi_{A'} = 0$ now reads $\mathscr{D}_V \psi = 0$ for

$\psi \in \Gamma(\mathbb{M}^I, \mathcal{S}_-)$. To describe this equation on $\mathbb{A}^I$ let $\mathcal{O}(1,0)$ (resp. $\mathcal{O}(0,1)$) denote the sheaf of holomorphic sections of the hyperplane section bundle on $\mathbb{P}$ (resp. $\mathbb{P}^*$) pulled back by projection to $\mathbb{P} \times \mathbb{P}^*$ and let $\mathcal{O}(p,q) = \mathcal{O}(1,0)^p \otimes \mathcal{O}(0,1)^q$. Equivalently, a section of $\mathcal{O}(p,q)$ is a holomorphic function $f(Z,W)$ homogeneous of degree p in Z and q in W. The Penrose transform gives

$$\mathcal{P}: H^1(\mathbb{A}^I, \mathcal{O}(L)(-3,0)) \to \ker \mathcal{D}_V: (\Gamma(\mathbb{M}^I, \mathcal{S}_-) \to \Gamma(\mathbb{M}^I, \mathcal{S}^+)). \quad (3.2\text{a})$$

Similarly, if $\mathcal{D}_V^* \leftrightarrow (\nabla^{AA'} - V^{AA'})$ then

$$\mathcal{P}: H^1(\mathbb{A}^I, \mathcal{O}(L^*)(0,-3)) \to \ker \mathcal{D}_V^*: \Gamma(\mathbb{M}^I, \mathcal{S}_-) \to \Gamma(\mathbb{M}^I, \mathcal{S}^+). \quad (3.2\text{b})$$

Let

$$\Psi \in H^1(\mathbb{A}^I, \mathcal{O}(L(-3,0))$$
$$\Phi \in H^1(\mathbb{A}^I, \mathcal{O}(L^*)(0,-3))$$

represent under (3.2) the fields $\psi_{A'}$ and ϕ_A of equations (3.1) (for $\lambda = 1$).

Part (c). As a special case of the general Yang–Mills case (Henkin & Manin [8]) the line bundle L extends automatically to a unique line bundle $L^{(2)}$ on $\mathbb{A}^{I(2)}$ and the obstruction $\omega(L) \in H^2(\mathbb{A}^I, \mathcal{O}(-3,-3))$. We note that $\mathrm{Hom}(L,L)$ is trivial and that $\mathcal{O}(N^*) = \mathcal{O}(-1,-1)$ represents the current, in this case d^*F, under

$$\mathcal{P}: H^2(\mathbb{A}^I, \mathcal{O}(-3,-3)) \to \ker d: \Gamma(\mathbb{M}^I, \Omega^3) \to \Gamma(\mathbb{M}^I, \Omega^4).$$

The last equation of (2.1), $d^*F = \phi_A \psi_{A'} dz^{AA'}$ becomes

$$\omega(L) = \Phi \cup \Psi$$

for the cup product $\Phi \cup \Psi \in H^2(\mathbb{A}^I, \mathcal{O}(-3,-3))$.

To recapitulate, the Penrose transform gives a one-to-one correspondence between solutions of (2.1) and any set L, Ψ, Φ satisfying:

$$L \to \mathbb{A}^I \text{ is a holomorphic line bundle on } \mathbb{A}^I \text{ such that } L|_{\tau^{-1}(x)} \text{ is trivial for all } x \in \mathbb{M}^I, \quad (3.3\text{a})$$

$$\Psi \in H^1(\mathbb{A}^I, \mathcal{O}(L)(-3,0)), \quad (3.3\text{b})$$

$$\Phi \in H^1(\mathbb{A}^I, \mathcal{O}(L^*)(0,-3)), \quad (3.3\text{c})$$

$$\omega(L) = \Phi \cup \Psi \in H^2(\mathbb{A}^I, \mathcal{O}(-3,-3)). \quad (3.3\text{d})$$

Here, $\tau^{-1}(x)$ is the quadric $\sigma \circ \rho^{-1}(x)$ in $\mathbb{A}$ for $x \in \mathbb{M}$. The Penrose transform of any set $\{L, \Psi, \Phi\}$ satisfying (3.3) will yield a solution of (2.1), and all solutions arise this way.

4. The Inverse Penrose Transform of the Plane-Wave Solutions

We want to describe the inverse Penrose transform of the triple

$$\{\phi_A, \psi_{A'}, \phi_A \psi_{A'}(n \cdot z)^2\}$$

given in (3.1). Let us choose coordinate systems for our twistor space as follows:

$$(\omega^A, \pi_{A'}) = Z^\alpha, \qquad \text{homogeneous coordinates for } \mathbb{P}^I,$$
$$(\eta_A, \xi^{A'}) = W_\alpha, \qquad \text{homogeneous coordinates for } \mathbb{P}^{*I},$$
$$(z^{AA'}, w^{AA'}) \qquad \text{affine coordinates for } \mathbb{M}^I \times \mathbb{M}^I,$$

while

$$x^{AA'} = \tfrac{1}{2}(z^{AA'} + w^{AA'}),$$
$$y_{AA'} = \tfrac{1}{2}(z^{AA'} - w^{AA'})$$

are tangential and normal coordinates for the embedding

$$\mathbb{M}^I \to \mathbb{M}^I \times \mathbb{M}^I.$$

Define open coverings of $\mathbb{P}^I$ and $\mathbb{P}^{*I}$ by

$$V_0 = \{\pi_{0'} \neq 0\}, \qquad V_1 = \{\pi_{1'} \neq 0\}$$

and

$$V_0^* = \{\eta_0 \neq 0\}, \qquad V_1^* = \{\eta_1 \neq 0\},$$

respectively. Let

$$\tilde{U}_0 = V_0 \times V_0^*, \qquad \tilde{U}_2 = V_0 \times V_1^*,$$
$$\tilde{U}_1 = V_1 \times V_0^*, \qquad \tilde{U}_3 = V_1 \times V_1^*,$$

be a covering of $\mathbb{P}^I \times \mathbb{P}^{*I}$, and let $U_\alpha = \tilde{U}_\alpha \cap \mathbb{A}^I$ be the restriction of the covering to the affine ambitwistor space.

We begin by describing the bundle $L \to \mathbb{A}^I$ which corresponds to the potential

$$V = V_{AA'}\, dx^{AA'} = \phi_A \psi_{A'} (n \cdot x)^2\, dx^{AA'}$$

on the trivial line bundle over $\mathbb{M}_V$, and then its unique second-order extension $L^{(2)}$ and the obstruction to third-order extension $\omega(L)$. Define functions $c_\alpha : U_\alpha \to \mathbb{M}^I$ by specifying the components $\{c_\alpha^{AA'}\}$ by

$$\omega^A = ic_\alpha^{AA'}\pi_{A'},$$
$$\xi^{A'} = -ic_\alpha^{AA'}\eta_A,$$
$$c_0^{11'} = c_1^{10'} = c_2^{01'} = c_3^{01'} = 0.$$

The first two lines guarantee that $c_\alpha(X)$ lies on the null line $\tau(X)$ for every $X \in U_\alpha$, and the third picks out a particular point on $\tau(X)$ for each X. These functions $c^{AA'}$ are explicit rational functions of the coordiates of $\mathbb{P}^I \times \mathbb{P}^{*I}$ and depend only on the geometry of the manifolds involved, not on the fields or vector bundles being considered.

Now we define

$$f_{\alpha\beta}(X) = \int_{\gamma_X} V_{AA'}\,dx^{AA'}, \qquad X \in U_\alpha \cap U_\beta, \tag{4.1}$$

where γ_X is any path from $c_\alpha(X)$ to $c_\beta(X)$ which lies in $\tau(X)$, the functions $g_{\alpha\beta} = e^{f_{\alpha\beta}}$ give the transition functions for L. Explicitly, one carries out the integration and finds that

$$\begin{aligned} f_{\alpha\beta} &= \tfrac{1}{3}\phi_A\psi_{A'}(c_\beta^{AA'} - c_\alpha^{AA'})[(n\cdot c_\alpha)^2 + (n\cdot c_\alpha)(n\cdot c_\beta) + (n\cdot c_\beta)^2] \\ &= \tfrac{1}{3}(\phi\cdot\eta)(\psi\cdot\pi)(n\cdot\eta\pi)^{-1}[(n\cdot c_\beta)^3 - (n\cdot c_\alpha)^3]. \end{aligned}$$

Conversely, given such $f_{\alpha\beta}$ defining the line bundle L, we can obtain the potential $V_{AA'}$ as follows. We first split $f_{\alpha\beta}$ over $\mathbb{G}^I$ relative to the cover $\{\sigma^{-1}U_\alpha\}$ by

$$\sigma^* f_{\alpha\beta} = f_\alpha - f_\beta,$$

where each f_α is holomorphic on $\sigma^{-1}U_\alpha$; one such splitting is given by

$$f_\alpha = \tfrac{1}{3}(\phi\cdot\eta)(\psi\cdot\pi)(n\cdot\eta\pi)^{-1}[(n\cdot x)^3 - (n\cdot c_\alpha)^3].$$

We then solve the equation

$$\eta^A\pi^{A'}\nabla_{AA'}f_\alpha = \eta^A\pi^{A'}V_{AA'}$$

for $V_{AA'}$, obtaining

$$V_{AA'} = \phi_A\psi_{A'}(n\cdot x)^2.$$

Notice that a different choice of splitting would amount to only a different choice of gauge, that is, would modify the one-form V by an exact one-form, and so leads to an equivalent physical field.

The unique (up to isomorphism) extension of $L \to \mathbb{A}^I$ into $\mathbb{P}^I \times \mathbb{P}^{*I}$ can be constructed by a method inspired by the technique used in Isenberg *et al.* [9] to show that a third-order extension of L exists if and only if $d^*F = 0$. In that work the extensions of $L \to \mathbb{A}^I$ to formal neighborhoods of $\mathbb{A}^I$ in $\mathbb{P}^I \times \mathbb{P}^{*I}$ were related to extensions of V to one-forms defined on $\mathbb{M}^I \times \mathbb{M}^I$. Here we shall construct extensions of $f_{\alpha\beta}$ to functions on open sets in $\mathbb{P}^I \times \mathbb{P}^{*I}$ by a formula analogous to (4.1), replacing $V_{AA'}\,dz^{AA'}$ with a one-form on $\mathbb{M}^I \times \mathbb{M}^I$, and γ_X with a path in $\mathbb{M}^I \times \mathbb{M}^I$.

Consider $\mathbb{M}^I \times \mathbb{M}^I$ with coordinates as described above. Define functions

$$k_\alpha = (k_\alpha^+, k_\alpha^-) : \tilde{U}_\alpha \to \mathbb{M}^I \times \mathbb{M}^I$$

by demanding that

$$\omega^A = ik_\alpha^{-AA'}\pi_{A'}, \qquad \xi^{A'} = -ik_\alpha^{+AA'}\eta_{A'}$$

and also

$$\begin{array}{llll} k_0^{-01'} = i\xi^{1'}/\eta_0, & k_0^{-11'} = 0, & k_0^{+10'} = -i\omega^1/\pi_{0'}, & k_0^{+11'} = 0, \\ k_1^{-00'} = i\xi^{0'}/\eta_0, & k_1^{-10'} = 0, & k_1^{+10'} = 0, & k_1^{+11'} = -i\omega^1/\pi_{1'}, \\ k_2^{-01'} = 0, & k_2^{-11'} = i\xi^{1'}/\eta_1, & k_2^{+00'} = -i\omega^0/\pi_{0'}, & k_2^{+01'} = 0, \\ k_3^{-00'} = 0, & k_3^{-01'} = i\xi^{0'}/\eta_1, & k_3^{+00'} = 0, & k_3^{+01'} = -i\omega^0/\pi_{1'}. \end{array}$$

Then

$$k_\alpha(Z, W) \in \mathbf{P}_Z \times \mathbf{P}_W,$$

where $\mathbf{P}_Z, \mathbf{P}_W$ are the α-plane and β-plane corresponding to $E \in \mathbb{P}^I$, $W \in \mathbb{P}^{*I}$. In, particular, the functions $k_\alpha^\pm$ have been constructed to be extensions of the c_α: if $X = (Z, W) \in \mathbb{A}^I$, then $k_\alpha^+(X) = k_\alpha^-(X) = c_\alpha(X)$, so that

$$k_\alpha|_{\mathbb{A}^I} : \mathbb{A}^I \to \mathbb{M}^I = (\text{diagonal in } \mathbb{M}^I \times \mathbb{M}^I).$$

We extend the potential $V = V_{AA'}\,dz^{AA'}$ into $\mathbb{M}^I \times \mathbb{M}^I$ by defining

$$\tilde{V} = V_{AA'}^-\,dz^{AA'} + V_{AA'}^+\,dw^{AA'},$$

where $V_{AA'}^\pm$ are defined as follows. Let

$$P_{AA'BB'} = n_{AA'}\phi_B\psi_{B'} - n_{BB'}\phi_A\psi_{A'},$$

so that

$$F_{AA'BB'} = \frac{1}{2}\left(\frac{\partial V_{BB'}}{\partial x^{AA'}} - \frac{\partial V_{AA'}}{\partial x^{BB'}}\right) = P_{AA'BB'}(n \cdot x),$$

and then define

$$V^{\pm}_{AA'} = \tfrac{1}{2}\{\phi_A \psi_{A'}(n \cdot x)^2 \pm (P_{AA'BB'} + 2P_{AB'BA'})(n \cdot x) y^{BB'} - [\tfrac{1}{2}(P_{AA'BB'} n_{CC'} + P_{AA'CC'} n_{BB'}) + \tfrac{1}{3}(P_{AB'BA'} n_{CC'} + P_{AC'CA'} n_{BB'})] y^{BB'} y^{CC'}\}. \quad (4.2)$$

It is clear that upon restricting to the diagonal $\{y^{AA'} = 0\}$ we have

$$\tilde{V}|_{\mathbb{M}^I} = V.$$

One also checks (cf. Isenberg *et al.* [9]) that

$$\pi^{A'}\pi^{B'}\left(\frac{\partial V^-_{BB'}}{\partial z^{AA'}} - \frac{\partial V^-_{AA'}}{\partial z^{BB'}}\right) = O(y), \quad (4.3a)$$

$$\eta^A \eta^B \left(\frac{\partial V^+_{BB'}}{\partial w^{AA'}} - \frac{\partial V^+_{AA'}}{\partial w^{BB'}}\right) = O(y), \quad (4.3b)$$

$$\left(\frac{\partial V^-_{BB'}}{\partial w^{AA'}} - \frac{\partial V^+_{AA'}}{\partial z^{BB'}}\right) = O(y), \quad (4.3c)$$

and, moreover, modulo terms involving $d{*}F = {*}J$, all of these expressions vanish to order $O(y^2)$.

Remark. In Isenberg *et al.* [9] it is assumed that $d{*}F = 0$, and an explicit formula is given for constructing $V^{\pm}_{AA'}$ which satisfy Eqs. (4.3) to order $O(y^2)$. The expression (4.2) is derived from this general formula.

We are now in a position to define an extension of $f_{\alpha\beta}$ into open sets of $\mathbb{P}^I \times \mathbb{P}^{*I}$. Define on $\tilde{U}_\alpha \cap \tilde{U}_\beta$,

$$\tilde{f}_{\alpha\beta}(Z, W) = \int_{\gamma(Z,W)} V^-_{AA'}\, dz^{AA'} + V^+_{AA'}\, dw^{AA'},$$

where $\gamma(Z, W)$ is the straight line path from $k_\alpha(Z, W)$ to $k_\beta(Z, W)$ which lies in $\mathbf{P}_Z \times \mathbf{P}_W$. In the case where $X = (Z, W) \in \mathbb{A}^I$, then $\gamma(Z, W)$ lies in $\tau(X)$ which in turn lies in the diagonal $\mathbb{M}^I$, so that

$$\tilde{f}_{\alpha\beta}(Z, W) = \int_{\gamma(Z,W)} \tilde{V} = \int_{\gamma(Z,W)} V = f_{\alpha\beta}(X),$$

and $\tilde{f}_{\alpha\beta}|_{\mathbb{A}^I} = f_{\alpha\beta}$, as desired. We note that here, unlike the case for $f_{\alpha\beta}$, it is necessary to choose a particular path since varying the path will change $\tilde{f}_{\alpha\beta}$.

Since we are seeking a second-order extension of L, we desire that

$$\tilde{f}_{\alpha\beta} + \tilde{f}_{\beta\gamma} + \tilde{f}_{\gamma\alpha} = O(t_\alpha^3) \qquad \text{on } \tilde{U}_\alpha \cap \tilde{U}_\beta \cap \tilde{U}_\gamma, \quad (4.4)$$

where t_α is the normal coordinate of $U_\alpha \in \tilde{U}_\alpha$ satisfying $t_\alpha = (Z \cdot W)|_{U_\alpha}$. Indeed, in Isenberg *et al.* [9] certain choices were made which guaranteed that if Eqs. (4.3) were satisfied that this would be true. However, the above construction of $\tilde{f}_{\alpha\beta}$ from $\tilde{V}$ does not follow the same pattern, and one checks that we have only

$$\tilde{f}_{\alpha\beta} + \tilde{f}_{\beta\gamma} + \tilde{f}_{\gamma\alpha} = O(t_\alpha^2).$$

The two constructions are equivalent though, so we must be able to modify by a coboundary term so as to satisfy (4.4). That is, we look for functions $E_{\alpha\beta} \in \Gamma(\tilde{U}_\alpha \cap \tilde{U}_\beta, \mathcal{O})$ containing factors of t_α^2 such that if

$$F_{\alpha\beta} = \tilde{f}_{\alpha\beta} - E_{\alpha\beta},$$

then we have

$$F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha} = O(t_\alpha^3) \qquad \text{on } \tilde{U}_\alpha \cap \tilde{U}_\beta \cap \tilde{U}_\gamma,$$

and

$$F_{\alpha\beta}|_{U_\alpha \cap U_\beta} = f_{\alpha\beta}.$$

With the following choice of $\{E_{\alpha\beta}\}$ the functions $\{F_{\alpha\beta} = \tilde{f}_{\alpha\beta} - E_{\alpha\beta}\}$ define a second-order extension of L:

$$E_{01} = \frac{i}{12} t_0 t_1 \left\{ -3 \frac{\omega^1}{\pi_{0'}} \left[\phi_0 \psi_{0'} + \frac{\pi_{0'}}{\pi_{1'}} \phi_0 \psi_{1'} \right] \right\}$$

$$E_{02} = \frac{i}{12} t_0 t_2 \left\{ 3 \frac{\xi^{1'}}{\eta_0} \left[\phi_0 \psi_{0'} + \frac{\eta_0}{\eta_1} \phi_1 \psi_{0'} \right] \right\}$$

$$E_{03} = \frac{i}{12} t_0 t_3 \left\{ \frac{\pi_{1'} \xi^{1'} + \eta_0 \omega^0}{\eta_0 \pi_{0'}} \left[2\phi_0 \psi_{0'} + \frac{\pi_{0'}}{\pi_{1'}} \phi_0 \psi_{1'} + \frac{\eta_0}{\eta_1} \phi_1 \psi_{0'} + 2 \frac{\eta_0 \pi_{0'}}{\eta_1 \pi_{1'}} \phi_1 \psi_{1'} \right] \right\}$$

$$E_{12} = \frac{i}{12} t_1 t_2 \left\{ \frac{\pi_{1'} \xi^{1'} + \eta_1 \omega^1}{\eta_0 \pi_{0'}} \left[\phi_0 \psi_{0'} + 2 \frac{\pi_{0'}}{\pi_{1'}} \phi_0 \psi_{1'} + 2 \frac{\eta_0}{\eta_1} \phi_1 \psi_{0'} + \frac{\eta_0 \pi_{0'}}{\eta_1 \pi_{1'}} \phi_1 \psi_{1'} \right] \right\}$$

$$E_{13} = \frac{i}{12} t_1 t_3 \left\{ -3 \frac{\xi^{0'}}{\eta_0} \left[\phi_0 \psi_{1'} + \frac{\eta_0}{\eta_1} \phi_1 \psi_{1'} \right] \right\}$$

$$E_{23} = \frac{i}{12} t_2 t_3 \left\{ 3 \frac{\omega_1}{\pi_{0'}} \left[\phi_1 \psi_{0'} + \frac{\pi_{0'}}{\pi_{1'}} \phi_1 \psi_{1'} \right] \right\}.$$

The calculation of the obstruction to third-order extension is now straightforward. It is given by the cohomology class $\omega(L) = [\{G_{\alpha\beta}\}] \in H^2(\mathbb{A}^I, \mathcal{O}(-3, -3))$, where we define (up to coboundary)

$$G_{\alpha\beta\gamma} := F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha}.$$

We then obtain

$$G_{012} = G_{023} = 0,$$

$$G_{013} = -G_{123} = \frac{i}{6} t_0 t_1 t_2 \left\{ \phi_0 \psi_{0'} + \frac{\pi_{0'}}{\pi_{1'}} \phi_0 \psi_{1'} + \frac{\eta_0}{\eta_1} \phi_1 \psi_{0'} + \frac{\eta_0 \pi_{0'}}{\eta_1 \pi_{1'}} \phi_1 \psi_{1'} \right\}.$$

The basic result of Henkin and Manin [8] is that under the Penrose transform isomorphism

$$\mathscr{P} : H^2(\mathbb{A}^I, \mathcal{O}(-3, -3)) \xrightarrow{\cong} H^0(\mathbb{M}^I, \ker d : \Omega^3_{\mathbb{M}} \to \Omega^4_{\mathbb{M}}),$$

the obstruction class $\omega(L)$ maps to the axial current $*J = *(J_{AA'}\, dx^{AA'})$, where in this case $J_{AA'} = \phi_A \psi_{A'}$. That this is so can be seen explicitly in this case by considering the isomorphism (4.5) in some detail. If we write $H_{(a,b,c,d)}$ for the coefficient of the $(\pi_{0'})^a (\pi_{1'})^b (\eta_0)^c (\eta_1)^d$-term in the Laurent expansion of a holomorphic function H defined in the appropriate region, we see easily that

$$(G_{012} - G_{123})_{(-2,-1,-2,-1)} = \frac{i}{6} \phi_0 \psi_{0'} (Z \cdot W)^3 = \frac{i}{6} J_{00'} (Z \cdot W)^3,$$

$$(G_{012} - G_{123})_{(-1,-2,-2,-1)} = \frac{i}{6} \phi_0 \psi_{1'} (Z \cdot W)^3 = \frac{i}{6} J_{01'} (Z \cdot W)^3,$$

$$(G_{012} - G_{123})_{(-2,-1,-1,-2)} = \frac{i}{6} \phi_1 \psi_{0'} (Z \cdot W)^3 = \frac{i}{6} J_{10'} (Z \cdot W)^3,$$

$$(G_{012} - G_{123})_{(-1,-2,-1,-2)} = \frac{i}{6} \phi_1 \psi_{1'} (Z \cdot W)^3 = \frac{i}{6} J_{11'} (Z \cdot W)^3.$$

As seen in Pool [13] this implies that under (4.5) the cohomology class $[\{G_{\alpha\beta\gamma}\}]$ maps to $*(J_{AA'}\, dx^{AA'})$.

We note that we have actually extended a cohomology class and not a line bundle. The line bundle L defined by transition functions $g_{\alpha\beta} = e^{f_{\alpha\beta}}$ has extension $\tilde{L}$ defined by transition functions $\tilde{g}_{\alpha\beta} = e^{F_{\alpha\beta}}$. Then the obstruction $\omega(L)$ is defined by the cocycles

$$\begin{aligned} 1 - \tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha} &= 1 - e^{F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha}} \\ &= F_{\alpha\beta} + F_{\beta\gamma} + F_{\gamma\alpha} + O(t_\alpha^4), \end{aligned}$$

which define the same coclass in $H^2(\mathbb{A}^I, \mathcal{O}(-3, -3))$ as $\{G_{\alpha\beta\gamma}\}$.

We turn now to the description of the cohomology classes Ψ and Φ which correspond to the constant spinors $\psi_{A'}$ and ϕ_A under isomorphisms (3.2a) and (3.2b). For simplicity we shall assume that $\psi_{0'} = 0$, $\psi_{1'} = \psi$, and $\phi_0 = 0$, $\phi_1 = \phi$, so that

$$V = \phi\psi(n \cdot x)^2 \, dx^{11'},$$

and

$$J = \phi\psi \, dx^{11'};$$

this is equivalent to the general case via a change of coordinates on $\mathbb{T}$. To describe Ψ and Φ we will describe cocycles

$$\{\Psi_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}(L)(-3, 0))\}$$

and

$$\{\Phi_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}(L^*)(0, -3))\},$$

where we set $\Psi = [\{\Psi_{\alpha\beta}\}]$ and $\Phi = [\{\Phi_{\alpha\beta}\}]$. In particular, we will describe cocycles on $\mathbb{G}^I$ which push down to $\{\Psi_{\alpha\beta}\}$ and $\{\Phi_{\alpha\beta}\}$ on $\mathbb{A}^I$. To work on $\mathbb{G}^I$ we choose a cover $\{U'_\alpha = \sigma^{-1}(U_\alpha)\}$ and split the transition functions for σ^*L relative to this cover:

$$\sigma^* g_{\alpha\beta} = g_\alpha g_\beta^{-1} = e^{f_\alpha - f_\beta}.$$

The isomorphism (3.2a) can be expanded as (cf. Pool [13]):

$$\begin{aligned} H^1(\mathbb{A}^I, \mathcal{O}(L)(-3, 0)) &\cong H^1(\mathbb{G}^I, \sigma^{-1}\mathcal{O}(L)(-3, 0)) \\ &\cong H^1(\mathbb{G}^I, \ker d_\sigma : \mathcal{O}(\sigma^*L)(-3, 0) \to \Omega^1_\sigma(L)(-3, 0)) \\ &\cong H^0(\mathbb{M}^I, \ker \mathcal{D}_V : \mathcal{S}_- \to \mathcal{S}^+). \qquad (4.6) \end{aligned}$$

We will follow this chain one step at a time, from $\mathbb{G}^I$ to $\mathbb{M}^I$, and then from $\mathbb{G}^I$ to $\mathbb{A}^I$. Define functions $\zeta_{\alpha\beta} \in H^0(U'_\alpha \cap U'_{\beta'}, \mathcal{O}(-3, 0))$ by

$$\zeta_{01} = \zeta_{03} = -\zeta_{12} = \zeta_{23} = \psi/[\pi_{0'}(\pi_{1'})^2], \qquad \zeta_{02} = \zeta_{13} = 0.$$

We define functions $\tilde{\Psi}'_{\alpha\beta} \in H^0(U'_\alpha \cap U'_\beta, \mathcal{O}(\sigma^*L)(-3, 0))$ by specifying the trivializations $\{{}^\gamma\tilde{\Psi}_{\alpha\beta} :=$ γ-trivialization of $\tilde{\Psi}'_{\alpha\beta}\}$:

$${}^\gamma\tilde{\Psi}_{\alpha\beta} = g_\gamma \, {}^\gamma\zeta_{\alpha\beta},$$

where ${}^\gamma\zeta_{\alpha\beta}$ is the γ-trivialization of the section $\zeta_{\alpha\beta}$. If $\{h_{\alpha\beta}\}$ are the transition functions for $\mathcal{O}(-3, 0)$, we see that

$${}^\gamma\tilde{\Psi}_{\alpha\beta} = (g_{\gamma\sigma} h_{\gamma\sigma}) \tilde{\Psi}_{\alpha\beta},$$

so that $\{\tilde{\Psi}'_{\alpha\beta}\}$ defines a coclass in $H^1(\mathbb{G}^I, \mathcal{O}(\sigma^*L)(-3,0))$. One can check explicitly by integration over the fibre (using, for instance, the original contour integral formulation of Penrose, see Penrose & MacCallum [12]), that under the isomorphism

$$H^1(\mathbb{G}^I, \mathcal{O}(\sigma^*L)(-3,0)) \xrightarrow{\cong} H^0(\mathbb{M}^I, \mathcal{S}_-),$$

$[\{\tilde{\Psi}'_{\alpha\beta}\}]$ maps to $\psi_{A'}$, where $\psi_{0'} = 0$, $\psi_{1'} = \psi$.

We now modify $\{\tilde{\Psi}'_{\alpha\beta}\}$ by a coboundary $\{\kappa_\beta - \kappa_\alpha\}$ to obtain cocycles $\{\Psi'_{\alpha\beta}\}$ in the kernel of d_σ; the image of $[\{\Psi'_{\alpha\beta}\}]$ under the third of the isomorphisms (4.6) will still be $\psi_{A'}$. Thus we look for $\{\kappa_\alpha \in H^0(U'_\alpha, \mathcal{O}(\sigma^*L)(-3,0))\}$ such that

$$\Psi'_{\alpha\beta} := \tilde{\Psi}'_{\alpha\beta} + (\kappa_\alpha - \kappa_\beta)$$

satisfies $d_\sigma \Psi'_{\alpha\beta} = \eta^A \pi^{A'} \nabla_{AA'} \Psi'_{\alpha\beta} = 0$ on $U'_\alpha \cap U'_\beta$. One solution is to set $\kappa_0 = \kappa_2 = 0$ and choose κ_1 and κ_3 so that their trivializations satisfy

$$\eta^A \pi^{A'} \nabla_{AA'}\, {}^1\kappa_1 = \phi \psi^2 (n \cdot x)^2 \eta_0 \pi_{1'} g_1,$$

$$\eta^A \pi^{A'} \nabla_{AA'}\, {}^3\kappa_3 = \phi \psi^2 (n \cdot x)^2 \eta_0 \pi_{1'} g_3.$$

Integration in the fashion of (4.1) then gives ${}^1\kappa_1$ and ${}^3\kappa_3$ explicitly in terms of the rational functions c_α. On the cocycle level the second of the isomorphism mappings of (4.6) is just an injection, so $d_\sigma \Psi'_{\alpha\beta} = 0$ implies that $\{\Psi'_{\alpha\beta}\}$ defines an element of $H^1(\mathbb{G}^I, \sigma^{-1}\mathcal{O}(L)(-3,0))$. Finally, since these cocycles are constant along the fibres of $\sigma\colon \mathbb{G}^I \to \mathbb{A}^I$, we may push them down to $\mathbb{A}^I$ to obtain the desired coclass

$$[\{\Psi_{\alpha\beta}\}] \in H^1(\mathbb{A}^I, \mathcal{O}(L)(-3,0)),$$

where $\Psi'_{\alpha\beta} = \sigma^* \Psi_{\alpha\beta}$. It follows that $\{\Psi_{\alpha\beta}\}$ are then defined explicitly in terms of a polynomial in $[\pi_{0'}(\pi_{1'})^2]^{-1}$ and the rational functions c_α.

The construction of Φ follows the same lines. In this case, set

$${}^\gamma\tilde{\Phi}_{\alpha\beta} = g_\gamma^{-1}\, {}^\gamma\xi_{\alpha\beta},$$

where $\xi_{02} = \xi_{13} = \xi_{03} = \xi_{12} = \phi/[\eta_0(\eta_1)^2]$, $\xi_{01} = \xi_{23} = 0$, and find $\lambda_\alpha \in H^0(U'_\alpha, \mathcal{O}(L^*)(0,-3))$ such that

$$\Phi'_{\alpha\beta} = \tilde{\Phi}'_{\alpha\beta} + (\lambda_\alpha - \lambda_\beta)$$

satisfies $\eta^A \pi^{A'} \nabla_{AA'} \Phi'_{\alpha\beta} = 0$ on $U'_\alpha \cap U'_\beta$. One solution is $\lambda_0 = \lambda_1 = 0$ and choose λ_2, λ_3 to satisfy

$$\eta^A \pi^{A'} \nabla_{AA'}\, {}^2\lambda_2 = -\phi^2 \psi (n \cdot x)^2 \eta_1 \pi_{0'} g_2^{-1} \qquad \text{on } U'_2,$$

$$\eta^A \pi^{A'} \nabla_{AA'}\, {}^3\lambda_3 = -\phi^2 \psi (n \cdot x)^2 \eta_1 \pi_{0'} g_3^{-1} \qquad \text{on } U'_3.$$

It is then clear from comparison of the $(-1,-2,-1,-2)$-terms in their Laurent expansions that $\Phi \cup \Psi$ agrees with $\omega(L)$ in this special case.

It is interesting to consider the dependence of the solutions on the parameter in the original set of Eqs. (2.1),

$$\begin{aligned} (\nabla^{AA'} - \lambda V^{AA'})\phi_A &= 0, \\ (\nabla^{AA'} + \lambda V^{AA'})\psi_{A'} &= 0, \\ d^*F = J &= \lambda\phi_A\psi_{A'}\,dx^{AA'}, \qquad \lambda \in \mathbf{C}. \end{aligned} \tag{4.7}$$

These equations decouple as $\lambda \to 0$. By inspection we have the solution

$$V_{AA'} = \lambda\phi_A\psi_{A'}(n \cdot x)^2,$$

where ϕ_A, $\psi_{A'}$, and $n_{AA'}$ are as before. The transition functions for the related bundle $L_\lambda \to \mathbb{A}^I$ are just $e^{\lambda f_{\alpha\beta}}$, and the obstruction to third-order extension $\omega(L_\lambda)$ is defined by the cocycle

$$1 - e^{\lambda G_{\alpha\beta\gamma}} = \lambda G_{\alpha\beta\gamma} + O(\lambda^2).$$

Now let us examine the obstruction more closely. Let Ξ, Z denote the cohomology classes in

$$H^1(\mathbb{A}^I, \mathcal{O}(0,-3)), \qquad H^1(\mathbb{A}^I, \mathcal{O}(-3,0)),$$

respectively, which correspond to the constant spinors ξ_A, $\zeta_{A'}$ on $\mathbb{M}^I$. In the special case $\xi_0 = 0$, $\xi_1 = \phi$, $\zeta_{0'} = 0$, $\zeta_{1'} = \psi$, Ξ and Z are defined by the cocycles $\{\xi_{\alpha\beta}\}$, $\{\zeta_{\alpha\beta}\}$ above, pushed down to $\mathbb{A}^I$. We have

$$\nabla^{AA'}\xi_A = \nabla^{AA'}\zeta_{A'} = 0,$$

so ξ_A, $\zeta_{A'}$ are solutions to (4.7), where $\lambda = 0$, $V_{AA'} = 0$. Comparing the product $\Xi \cup Z \in H^2(\mathbb{A}^I, \mathcal{O}(-3,-3))$ with $\Phi \cup \Psi$ one finds

$$\Xi \cup Z = \Phi \cup \Psi + O(\lambda).$$

Since $\omega(L_\lambda) = \lambda(\Phi \cup \Psi)$ this implies that

$$\omega(L_\lambda) = \lambda(\Xi \cup Z) + O(\lambda^2),$$

and thus we obtain

$$[\{G_{\alpha\beta\gamma}\}] = \Xi \cup Z + O(\lambda).$$

Therefore to first order in λ one can identify the third-order obstruction to extension of L_λ as the product of the cohomology classes on $\mathbb{A}^I$ corresponding to the spinors ϕ_A, $\psi_{A'}$ on a flat background.

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