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On the Local Holomorphic Hull of a Real Submanifold in Several Complex Variables*

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Introduction

In a classical paper [11], E. E. Levi introduced in $\mathbb{C}^2$ what is now called the Levi form for real C^∞ hypersurfaces. He proved that if the Levi form did not vanish at a point of a real hypersurface in $\mathbb{C}^2$, then any holomorphic function defined on a particular side of the surface could be analytically continued to the other side (see Hervé [9] for an exposition of this in $\mathbb{C}^n$ in modern notation). The analytic continuation is based upon the well known "Kontinuitätssatz" of Hartogs.

H. Lewy proved in [12] that if the Levi form did not vanish at a point of a real hypersurface in $\mathbb{C}^2$, then functions on the hypersurface satisfying the induced Cauchy-Riemann equations could be extended to one side of the hypersurface as holomorphic functions. A consequence of Lewy's result is the following: If S is a hypersurface in $\mathbb{C}^2$ whose Levi form at $p \in S$ is not zero, then for any neighborhood N_p of p in S each holomorphic function defined in some neighborhood of N_p can be analytically continued to some open set Q depending only on N_p. In a certain sense we might say that the "hull of holomorphy" of the hypersurface S near p must contain an open set.

We can ask the following general question: What real submanifolds $M^k \subset \mathbb{C}^n$ of real dimension $k \leqq 2n - 1$ have the property that all functions holomorphic on M^k near p are necessarily holomorphic in some larger set not lying on M^k? For $k < 2n - 1$, are there any such submanifolds? Let us make this question somewhat more precise in the following manner (see Section 1 for exact definitions). We shall say that a real k-dimensional submanifold $M^k \subset \mathbb{C}^n$ is locally extendible at $p \in M^k$ if, for any sufficiently small connected neighborhood B_p about p, there exists a connected set Q, $Q \neq \varnothing$, $\bar{Q} \cap B_p \cap M^k \neq \varnothing$, $Q \cap M^k = \varnothing$, such that each holomorphic function defined in a domain containing $B_p \cap M^k$ can be analytically continued to a holomorphic function defined in a domain containing $\bar{Q} \cup (B_p \cap M^k)$. The question becomes: when are submanifolds $M^k \subset \mathbb{C}^n$, $k \leqq 2n - 1$, locally extendible?

Lewy gives an example in [13] of an $M^4 \subset \mathbb{C}^3$ (and hence of real codimension

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2) which has the property that functions satisfying the induced Cauchy-Riemann equations on M^4 are boundary values of holomorphic functions in an open set Q. It can be shown that this same example is locally extendible in the sense defined above.

In this paper we attempt to characterize these examples of lower dimensional extension. In Section 1 we give precise definitions and some necessary background. We also introduce a Levi form which is defined for lower dimensional submanifolds as well as for hypersurfaces (cf. [17]). In Section 2 we prove a general result on analytic continuation involving families of analytic polydiscs (Theorem 1) which we use later. In Section 3 we generalize Lewy's result for hypersurfaces to $\mathbb{C}^n$, following his method of analysis. A proof of the same generalization, based upon a different method, is given by Hörmander in [10]. We prove a converse to this (Theorem 3) by showing that if the Levi form is identically zero near a point on the hypersurface, then the hypersurface is locally the countable intersection of domains of holomorphy. This implies that no extension is possible in this case.

In Section 4 we indicate the construction of analytic polydiscs whose distinguished boundary is required to lie on a given submanifold M^k. This follows the lines of Bishop's work on analytic discs, [4]. Using this construction we prove our main result (Theorem 5) which is a generalization of the local extendibility of a hypersurface at a point where the Levi form is not zero: Let M^k be a real k-dimensional C^∞ submanifold of $\mathbb{C}^n$ with $k > n$ and let $p \in M^k$. Suppose that the complex dimensions of the complex tangent spaces to M^k near p are constant and equal to $k - n$. If the Levi form of M^k is not zero at p, then M^k is locally extendible at p.

1. Preliminaries

Let $\mathbb{C}^n$ denote complex Euclidean space, $n > 1$. Suppose K is a connected subset of $\mathbb{C}^n$. We shall say that f is a holomorphic function on K if f is defined and holomorphic in some domain containing K (all holomorphic functions are to be single-valued; see [2], [3], [9], and [10] for background in several complex variables). Let $H(K)$ denote the class of all holomorphic functions on K. If $K' \supset K$, and K' is connected, then there is a natural mapping

$$H(K') \to H(K) .$$

If $K' - K \neq \varnothing$, we shall say that K is *extendible* to K' if this mapping is onto. This means that all holomorphic functions defined on K can be analytically continued (in a single-valued manner) to K'.

We shall consider differentiable real submanifolds of $\mathbb{C}^n$, where differentiable will always mean C^∞. Let M^k be a real k-dimensional differentiable regularly embedded submanifold of $\mathbb{C}^n$. Let T_p be the real tangent space to M^k at $p \in M^k$. The complex tangent space to M^k at p, C_p, is defined as the maximal complex subspace of T_p. We shall say that M^k is *locally extendible at* p if it has the following property: For any sufficiently small connected neighborhood N_p of p in M^k, there exists a connected set Q with $Q \neq \varnothing$, $\bar{Q} \cap N_p \neq \varnothing$, $Q \cap M^k = \varnothing$, such that N_p is extendible to $N_p \cup \bar{Q}$.

Let $m(q)$ be the complex dimension of C_q, and suppose that $m(q)$ is constant $(= m)$ near $p \in M^k$. In this case we shall say that M^k is *generic* at p. Then there are m infinitesimal transformations of $\mathbb{C}^n$ near p, $X_1, \cdots, X_m$, which generate C_q at each point $q \in M^k$ near p. Their conjugates $\bar{X}_1, \cdots, \bar{X}_m$, are locally defined infinitesimal transformations which are tangent to M^k near p. They induce vector fields on M^k and these are the induced Cauchy-Riemann operators on M^k near p. In Section 3 we shall write out explicitly the transformations we use when M^k is a real hypersurface in $\mathbb{C}^n$ (see Sommer [16]). Let N_p be a connected neighborhood of p in M^k. Let $C^\infty(N_p)$ denote the class of differentiable functions on N_p, where we mean here functions defined on the manifold M^k and differentiable in the local coordinates of M^k. We shall denote by

$$R(N_p) = \{f \in C^\infty(N_p) \colon \bar{X}_j f = 0, j = 1, \cdots, m\}$$

the class of *relatively holomorphic functions* on N_p. In Section 3 we shall investigate "extendibility" with respect to $R(N_p)$ in the case of a hypersurface.

We want to introduce a Levi form analogous to that introduced by Hermann [17] which is defined also for lower dimensional submanifolds in $\mathbb{C}^n$. We shall consider only generic submanifolds M^k near a point $p \in M^k$. If v is a vector in C_p, then there is a complex vector field X defined near p whose restriction to the point p, X_p, is v. Let $\bar{X}$ be the complex conjugate of X and let $[X, Y]$ be the commutator of the vector fields X and Y (see [16]). Let π denote the projection of T_p onto T_p/C_p. The Levi form L is defined as a mapping

$$L \colon C_p \to T_p/C_p$$

given by

$$L(v) = \pi[X, \bar{X}]_p .$$

This is independent of the choice of X. In Section 3 we see that, in the case of a hypersurface, $L(v)$ agrees with the usual quadratic form obtained from the complex Hessian acting on the complex tangent space (see [7], [15], and [16]). We shall say $L = 0$ at p if $L(v) = 0$ for all $v \in C_p$.

Let $U \Subset \mathbb{C}^n$ be a domain. U is a *domain of holomorphy* if there exists a function F which is holomorphic in U and singular at every boundary point of U (see [3]). It is clear that a domain of holomorphy cannot be extendible to any strictly larger set.

An S_δ is a relatively compact subset of $\mathbb{C}^n$ which can be written as the intersection of countably many domains of holomorphy (cf. Rossi [15]). We shall say that a set $K \Subset \mathbb{C}^n$ is *locally* S_δ at $p \in K$ if, for any sufficiently small ball B_p about p, $B_p \cap K$ is an S_δ. We remark that if K^k is locally S_δ at $p \in K^k$, then K^k is not locally extendible at p, and, conversely, if K^k is locally extendible at p, then K^k is not locally S_δ at p.

2. Analytic Continuation

Let D be the unit m-disc in $\mathbb{C}^m$ given by

$$D = \{\zeta = (\zeta_1, \cdots, \zeta_m) \colon |\zeta_j| < 1, j = 1, \cdots, m\}.$$

The *distinguished boundary* of D will be denoted by bD and is given by

$$bD = \{\zeta = (\zeta_1, \cdots, \zeta_m) \colon |\zeta_j| = 1, j = 1, \cdots, m\}.$$

An *analytic m-disc* A in $\mathbb{C}^n$ is the image of $\bar{D}$ under a continuous mapping

$$F \colon \bar{D} \to \mathbb{C}^n ,$$

where F is holomorphic on D. A is *nondegenerate* if F has maximal rank at each point of D and *completely degenerate* if F is a constant mapping (the image of $\bar{D}$ is a point in $\mathbb{C}^n$). bA will denote the distinguished boundary of A and is defined by

$$bA = F(bD) .$$

If f is holomorphic on A, then $g(\zeta) = f(F(\zeta))$ is a holomorphic function on D which assumes its maximum on bD (see [2]). We obtain therefore

$$\max_{A} |f| = \max_{bA} |f| ,$$

which is the maximum principle for analytic m-discs we shall need.

A sequence of analytic m-discs $A_j = F_j(D), j = 1, 2, \cdots$, converges uniformly to $A_0 = F_0(\bar{D})$ if we have $\|F_j(bD) - F_0(bD)\| \to 0$, as $j \to \infty$, in the maximum norm over bD. By the maximum principle, the analytic discs A_j converge to A_0 in the uniform norm on $\bar{D}$.

We state now part of the classical theorem of Cartan-Thullen, which we shall need in the proof of Theorem 1. Let K and B be domains in $\mathbb{C}^n$, $K \Subset B \Subset \mathbb{C}^n$, and let $\hat{K}$ denote the holomorphic hull of K with respect to $H(B)$, the class of holomorphic functions in B, i.e.,

$$\hat{K} = \{z \in B \colon |f(z)| \leqq \max_{\zeta \in K} |f(\zeta)|, f \in H(B)\} .$$

Let $\Delta_B(K)$ denote the minimal distance from K to the boundary of B, and let $Z(\zeta; \rho)$ denote the polydisc with center ζ and radius ρ.

THEOREM (Cartan-Thullen [5]). *If $\zeta \in \hat{K}$, then every function $f \in H(B)$ is holomorphic in the polydisc $Z(\zeta, r)$, where $r = \Delta_B(K)$.*

This theorem is proved by expansion in power series and is our basic analytical tool in the simultaneous analytic continuation of classes of holomorphic functions.

Let $T \subset \mathbb{R}^q$. Consider a continuous mapping

$$F \colon \bar{D} \times T \to \mathbb{C}^n$$

such that $F(\zeta, t)$ is holomorphic in ζ for t fixed in T. Then T parametrizes a family of analytic discs $A(t) = F(\bar{D}, t)$ in $\mathbb{C}^n$. We shall say that $\{A(t)\}$ is a *continuous family of analytic m-discs* if $A(t_j)$ converges uniformly to $A(t_0)$ as $t_j \to t_0$ in T. Set

$$A = F(\bar{D} \times T) ,$$

$$bA = F(bD \times T) .$$

Note that bA is not the full topological boundary of A. We then have the following theorem on simultaneous analytic continuation.

THEOREM 1. *Let A be a continuous family of analytic discs parametrized by a simply-connected and connected set $T \subset \mathbb{R}^q$. If for some $t_0 \in T$, $A(t_0)$ is completely degenerate, then bA is extendible to A; moreover, if f is holomorphic in a domain $B \supset bA$, then there is a domain $U \supset B$ such that $U \supset A$ and a holomorphic function g in U such that $g \mid U = f$.*

Proof: Suppose f is holomorphic in a domain $B \supset bA$. Let $t_0 \in T$ be such that $A(t_0)$ is a point in $\mathbb{C}^n$. Let γ be a path from t_0 to any other point $t \in T$ and let C_γ denote the corresponding one-parameter family of analytic discs in $\mathbb{C}^n$ given by $F(\bar{D} \times \gamma)$. Our given function f is defined in $B \supset A(t_0)$, since $A(t_0)$ is a point in $\mathbb{C}^n$, and we want to continue f analytically "along γ" to points in $A(t)$. Suppose γ is parametrized by $\gamma(s)$, $0 \leqq s \leqq 1$, with $\gamma(0) = t_0$, $\gamma(1) = t$. Let γ_τ denote the curve obtained by letting s vary from 0 to $\tau \leqq 1$. Let σ denote the set of those τ which have the property that there exists a domain U and a function $g \in H(U)$ such that $U \supset F(\bar{D} \times \gamma_\tau)$, $U \supset B$, and $g \mid B = f$. It is clear that σ is nonempty and open. We want to show that σ is also closed.

Let $\{\tau_j\}$ be a sequence in σ converging to τ_0, and set $A_j = A(\gamma(\tau_j))$, $j = 0, 1, 2, \cdots$. If $\tau_j \geqq \tau_0$ for some j, then $\tau_0 \in \sigma$. Thus we may assume that $\tau_j \to \tau_0$ monotonically, and moreover that there are domains $U_j \supset B$ such that $U_j \supset F(\bar{D} \times \gamma_{\tau_j})$, $U_{j+1} \supset U_j$, and there are functions $g_j \in H(U_j)$ with $g_j \mid B = f$. Let $U = \bigcup_{j=1}^{\infty} U_j$, and define $g \in H(U)$ by setting $g = g_j$ in U_j. This is well defined since in the (connected) intersection of two such defining domains, $U_i \cap U_j$, the functions g_i and g_j equal f on the domain $B \subset U_i \cap U_j$. U clearly contains the image of $\gamma(s)$, $0 \leqq s < \tau_0$, under the mapping F. We want to extend U by continuing g analytically so that it will contain A_0. Every point of A_0 is either a boundary or interior point of U, so in order to show that $\tau_0 \in \sigma$, it suffices to show that for each point p of A_0, g can be continued analytically to a neighborhood of p.

Let K be a neighborhood of bA_0 with $K \Subset U$, and set $d = \Delta_U(K)$. Let z^0 be a point in A_0. Since A_j converges uniformly to A_0, we can choose j large enough so that $bA_j \subset K$ and $z^0 \in Z(z, d)$ for some $z \in A_j$. If $h \in H(U)$, we have, by the maximum principle for analytic m-discs,

$$|h(z)| \leqq \max_{\zeta \in bA_j} |h(\zeta)| \leqq \max_{\zeta \in K} |h(\zeta)|,$$

from which it follows that $z \in \hat{K}$, the hull of K with respect to $H(U)$. By the Cartan-Thullen Theorem all functions in $H(U)$, and in particular g, can be continued analytically to $Z(q, d) \ni z^0$ (this last argument is due to Behnke-Sommer [1]).

Thus we can continue f analytically along γ to any point t in T. Since T is simply-connected, a monodromy argument will show that we can continue f analytically to all points of $F(\bar{D} \times T)$, and the remainder of the theorem then follows. Q.E.D.

3. Lewy's Hypersurface Theorem in $\mathbb{C}^n$

Let S be a real differentiable regularly embedded hypersurface in $\mathbb{C}^n$ and suppose $p \in S$. Then S is defined near p by

$$\phi(z_1, \cdots, z_n) = 0,$$

where ϕ is a real-valued differentiable function defined near p with $\phi(p) = 0$ and with a nonvanishing gradient at p. The Levi form L is defined invariantly near p. We want to show that $L(v)$ at p is equal to the quadratic form

$$\bar{v}Hv,$$

where $H = (\phi_{z_i\bar{z}_j})$ is the complex Hessian matrix evaluated at p and $v \in C_p$, where C_p is the complex tangent space to S at p. Set

$$\phi_k = \phi_{z_k}, \qquad \phi_{\bar{k}} = \phi_{\bar{z}_k}. \qquad k = 1, \cdots, n,$$

and assume that $\phi_n \neq 0$ at p. Set

$$X_k = \phi_n \frac{\partial}{\delta z_k} - \phi_k \frac{\partial}{\partial z_n}, \qquad k = 1, \cdots, n-1,$$

$$\bar{X}_k = \phi_{\bar{n}} \frac{\partial}{\partial \bar{z}_k} - \phi_{\bar{k}} \frac{\partial}{\partial \bar{z}_n}, \qquad k = 1, \cdots, n-1,$$

$$X_n = \phi_{\bar{n}} \frac{\partial}{\partial z_n} - \phi_n \frac{\partial}{\partial \bar{z}_n}.$$

The set $\{X_1, \cdots, X_{n-1}\}$ spans the complex tangent spaces near p, and $\{X_1, \cdots, X_{n-1}, \bar{X}_1, \cdots, \bar{X}_{n-1}, X_n\}$ spans the real tangent spaces near p (see [16]). We have

$$[X_i, \bar{X}_j] = \sum_{k=1}^{n-1} (a_k X_k + b_k \bar{X}_k) - c_{ij} X_n,$$

for $i, j = 1, \cdots, n-1$, where the coefficients $\{a_k\}$ and $\{b_k\}$ depend on i, j and are given explicitly by Sommer [16]. Sommer shows that the c_{ij} are the coefficients of the Hermitian form $\bar{v}Hv$ under the side condition that $v \in C_p$. If $v \in C_p$ and $v = X_p$, where X is a locally defined complex vector field near p, then

$$X = \sum_{k=1}^{n-1} d_k X_k,$$

where the d_k are C^∞ functions defined near p. We have

$$[X, \bar{X}] = \sum_{i,j=1}^{n-1} d_i \bar{d}_j [X_i, \bar{X}_j].$$

Using the above expression for the commutators, evaluating the vector fields at p, and projecting into the quotient space T_p/C_p, we obtain

$$\pi[X, \bar{X}]_p = \sum_{i,j=1}^{n-1} c_{ij}\, d_i\, \bar{d}_j$$

which is $\bar{v}Hv$, as indicated above. Therefore, $L = 0$ at $p \Leftrightarrow H$ has zero eigenvalues in C_p. In $\mathbb{C}^2$ there is only one coefficient c_{11} and this is the classical Levi determinant ([2], p. 29).

LEMMA 1. *Suppose $L \neq 0$ at p, then there is a holomorphic change of coordinates such that in the new coordinates, $z_j = x_j + iy_j$, $j = 1, \cdots, n$, $p = (0, \cdots, 0)$, and moreover:*

$$\text{(1)} \qquad \begin{aligned} \phi_{x_m}|_p &= 0\,, \qquad && m = 1, \cdots, n\,, \\ \phi_{y_m}|_p &= 0\,, && m = 1, \cdots, n-1\,, \\ \phi_{y_n}|_p &\neq 0\,, \end{aligned}$$

$$\text{(2)} \qquad (\phi_{x_1x_1}\phi_{y_1y_1} - \phi^2_{x_1y_1})|_p > 0\,.$$

Proof: Without loss of generality we can assume (after a complex-linear change of coordinates) that $p = (0, \cdots, 0)$, that we have complex coordinates $z_j = x_j + iy_j$, $j = 1, \cdots, n$, that S is given by $[\phi(z) = 0]$, and that the real tangent plane to S at p, T_p, is given by $[y_n = 0]$. Hence, in these coordinates, (1) is satisfied. We need to find a suitable new set of coordinates so that (2) is also satisfied, without changing (1).

$L \neq 0$ at p means that the Hessian matrix H at p has an eigenvector $v = (a_1, \cdots, a_{n-1}, 0) \neq 0$ in C_p corresponding to an eigenvalue $\lambda \neq 0$. Let A be a nonsingular $n \times n$ matrix with v as its first column and with its last row equal to $(0, \cdots, 0, 1)$. Then we can define the complex-linear change of coordinates in $\mathbb{C}^n$, $\zeta = Az$, where $\zeta = (\zeta_1, \cdots, \zeta_n)$, $\zeta_j = \xi_j + i\eta_j$. Moreover, we can define $\psi(\zeta) = \phi(A^{-1}\zeta)$ and hence $S = [\psi(\zeta) = 0]$. We obtain by a calculation

$$4\psi_{\zeta_1\bar{\zeta}_1}|_p = (\psi_{\xi_1\xi_1} + \psi_{\eta_1\eta_1})|_p \neq 0\,.$$

In the ζ coordinates, T_p and C_p are still given by $[\eta_n = 0]$ and $[\zeta_n = 0]$, respectively.

Following Lewy [12], we define new coordinates $\tau_j = s_j + it_j$ by the quadratic transformation

$$\begin{aligned} \tau_n &= \zeta_n + \tfrac{1}{2}\beta\zeta_1^2\,, \\ \tau_j &= \zeta_j\,, \qquad && j = 1, \cdots, n-1\,, \end{aligned}$$

where $\beta = \beta_1 + i\beta_2$ is a suitable complex constant depending on ψ to be chosen. Set $\chi(\tau) = \psi(\zeta(\tau))$. T_p and C_p are still given by $[t_n = 0]$ and $[\tau_n = 0]$, respectively, as before. By calculating the second derivatives of χ in terms of the derivatives of ψ and choosing β such that $\chi_{s_1s_1} - \chi_{t_1t_1} = \chi_{s_1t_1} = 0$ at p we obtain easily (see [12])

$$4(\chi_{s_1s_1}\chi_{t_1t_1} - \chi^2_{s_1t_1}) = (\psi_{\xi_1\xi_1} + \psi_{\eta_1\eta_1})^2 > 0\,.$$

Lewy has shown in [12] that if S is strongly pseudoconvex at $p \in S$ in $\mathbb{C}^2$ (i.e., the Levi determinant is not equal to zero), then S has the following property of extension. For any neighborhood N_p of p in S there is an open set Q (a neighborhood of p on one side of S) such that all holomorphic functions on N_p can be continued analytically to Q. Lewy proves a stronger result by only requiring the functions he extends to be defined on the hypersurface, satisfying the induced Cauchy-Riemann equations. We generalize these results to $\mathbb{C}^n$, following the approach of Lewy.

THEOREM 2. *Suppose $L \neq 0$ at $p \in S$; then, given any connected neighborhood N_p of p in S, there exists a domain $Q \subset \mathbb{C}^n$, depending only on N_p, with the following properties:*

(a) $Q \neq \varnothing$, $Q \cap S = \varnothing$, $\bar{Q} \cap N_p$ *is a neighborhood of p in S.*

(b) N_p *is extendible to* $N_p \cup \bar{Q}$.

(c) *If $f \in R(N_p)$, then there exists a function $\tilde{f} \in H(Q)$ such that $\tilde{f}$ is continuous on $\bar{Q}$ and*

$$\tilde{f} \mid \bar{Q} \cap N_p = f.$$

Remark. (b) follows from (c), but we prove (b) directly using Theorem 1, since this is the type of analytic continuation which we are able to generalize to lower dimensional submanifolds.

Proof: By Lemma 1 we may assume that we have coordinates $z_j = x_j + iy_j$, $j = 1, \cdots, n$, so that (1) and (2) hold. Since $\phi(0) = 0$, and $\phi_{y_n} \neq 0$ at p, we may solve for y_n and obtain

$$y_n = \psi(z_1, \cdots, z_{n-1}, x_n),$$

where the first derivatives of ψ all vanish at p.

Consider the system of equations

$$\begin{aligned} \psi_{x_1}(z_1, \cdots, z_{n-1}, x_n) &= 0, \\ \psi_{y_1}(z_1, \cdots, z_{n-1}, x_n) &= 0, \\ \psi(z_1, \cdots, z_{n-1}, x_n) - y_n &= 0. \end{aligned} \tag{3}$$

The Jacobian determinant of this system with respect to the variables (x_1, y_1, y_n) is not zero at p by (2), and $z = 0$ satisfies this system by (1). Hence we can solve for these three variables near $z = 0$ and obtain

$$\begin{aligned} x_1 &= x_1(z_2, \cdots, z_{n-1}, x_n) = x_1(t), \\ y_1 &= y_1(z_2, \cdots, z_{n-1}, x_n) = y_1(t), \\ y_n &= \psi(x_1(t), y_1(t), t) = y_n(t), \end{aligned} \tag{4}$$

where we have set $(z_2, \cdots, z_{n-1}, x_n) = t$. Let V be the real submanifold defined by (4). Since the Jacobian matrix of the system (6) has maximal rank 3 at p, V is a submanifold of S of real dimension $2n - 3$ which passes through the point p.

Let P be the complex subspace of $\mathbb{C}^n$ defined by $[z_1 = 0]$ with coordinates $w = (t, y_n) = (z_2, \cdots, z_n)$, and let P_0 be the real subspace of P defined by $[y_n = 0]$ with coordinates $t = (z_2, \cdots, z_{n-1}, x_n)$. A neighborhood of $t = 0$ in P_0, call it N, parametrizes V. Let V' be the projection of V onto P. This becomes a mapping of N into P given by $t \to (t, y_n(t))$ which is a hypersurface in P passing through the origin $w = 0$. For a given fixed $w \in P$, let M_w be the complex affine plane in $\mathbb{C}^n$ with coordinate z_1 given by fixing $(z_2, \cdots, z_n) = w$.

Define J_w by $J_w = M_w \cap S$. For $w = (t, y_n)$ we can investigate the nature of J_w by restricting ourselves to the real three-dimensional affine subspace B_t defined by fixing t in $\mathbb{C}^n$, i.e.,

$$B_t = \{z\colon (z_2, \cdots, z_{n-1}, x_n) = t\}$$

for fixed t. B_t contains M_w as an affine subspace translated by the parameter y_n. In B_t, S is a two-dimensional surface given by

$$y_n = \psi(x_1, y_1, t), \tag{5}$$

where t is now fixed. At $(x_1(t), y_1(t)) = z_1(t)$, ψ has a critical point, and ψ has either a local strong maximum or minimum there by virtue of (2). Without loss of generality we may assume that $\psi_{x_1 x_1} > 0$ near p and hence that ψ has a local strong minimum at the critical point. J_w is the solution of (5) for fixed y_n; $y_n(t)$ is the critical value of ψ at $z_1(t)$, and the point $(z_1(t), y_n(t))$ is the only solution to (5) for $y_n = y_n(t)$. For $y_n > y_n(t)$, we have regular values of ψ, and the solutions to (5) (the level curves) are differentiable submanifolds of B_t of one real dimension, and, in fact, are closed Jordan curves. For $y_n < y_n(t)$, there are no solutions since near a local minimum there are no values of the function less than the critical value. As t varies near $t = 0$, we obtain in this way a $(2n - 2)$-parameter family of such curves J_w.

We can find an $\varepsilon > 0$ so that each point of the closure $\overline{W}$ of the domain

$$W = \{w : |w| < \varepsilon, y_n > y_n(t)\}$$

determines a curve J_w as in the above construction which will degenerate into a point for $y_n = y_n(t)$. If N_p is given as an *a priori* neighborhood of p in S, we can choose ε small enough so that all of the curves J_w lie in N_p. This construction defines a mapping

$$F\colon bD \times \overline{W} \to \mathbb{C}^n,$$

where D is the unit disc in $\mathbb{C}$, and $F(bD \times w) = J_w$. It follows from the implicit function theorem and the compactness of bD that the curves J_w vary uniformly with w.

Each J_w lies in the complex plane M_w and hence bounds a simply connected domain D_w in M_w. By a well known theorem on conformal mapping (see [8]), the set

$$\{A_w = D_w \cup J_w\}$$

becomes a family of analytic discs which is continuous with respect to the parameter w. This extends the mapping F to the interior of D so that we have

$$F\colon \bar{D} \times W \to \mathbb{C}^n ,$$

where $F(\zeta, w)$ is holomorphic in ζ for fixed w, and $F(D \times w) = D_w$, $F(\bar{D} \times w) = A_w$. Define

$$A = F(\bar{D} \times \bar{W}) ,$$

$$Q = F(D \times W) .$$

We want to show that Q is open. Suppose $q \in Q$, then $q \in D_{w_0}$ for some $w_0 \in W$. We can find a simply connected domain U in M_{w_0} such that (i) $U \Subset D_{w_0}$, (ii) $q \in U$, and (iii) the critical point of ψ corresponding to $w_0 = (t^0, y_n^0)$ given by $z_1(t^0) = (x_1(t^0), y_1(t^0))$ lies in U', where we let K' in this argument denote the projection of a set $K \subset \mathbb{C}^n$ onto the z_1-plane. Let N be a neighborhood (in $\mathbb{C}^n$) of J_{w_0} such that $N' \cap U' = \varnothing$. We can then find a $d > 0$ such that for $|w - w_0| < d$ we have $J_w \subset N$ and the critical point $z_1(t)$ corresponding to $w = (t, y_n)$ lies in U'. This last assertion follows from the continuity of the critical points on V' with respect to the parameter t and the fact that

$$|t - t^0| \leqq |w - w_0| = |(t, y_n) - (t^0, y_n^0)| .$$

Set $B = \{w : |w - w_0| < d\}$. We then have

$$U' \times B \subset Q$$

if we know that for every $w \in B$, $U' \subset D'_w$. Therefore we can conclude that Q is open if we can prove this last statement in the z_1-plane. The critical point corresponding to the domain $D'_w (= z_1(t)$, where $w = (t, y_n))$, lies in both D'_w and U' by construction. Hence $D'_w \cap U' \neq \varnothing$, but

$$U' \cap bD'_w = U' \cap J'_w = \varnothing .$$

Since $U' \cup D'_w$ is a domain in $\mathbb{C}^1$ it is clear that $U' \subset D'_w$. Thus Q is open. A similar argument using the coordinates (z_1, t) parametrizing S shows that $\bar{Q} \cap N_p$ is a neighborhood of p in S. Hence part (a) of the theorem has been proven.

Part (b) follows immediately from the above construction and Theorem 1 (for 1-discs), where we take T to be $\bar{W}$ and t_0 to be a boundary point of W lying in V'.

All that remains to show is part (c), the relative holomorphic extension property. Suppose $f \in R(N_p)$. We can then define the tangential Cauchy-Riemann operators to be

$$\bar{X}_k = \phi_{\bar{n}} \frac{\partial}{\partial \bar{z}_k} - \phi_{\bar{k}} \frac{\partial}{\partial \bar{z}_n} , \qquad k = 1, \cdots, n-1 .$$

We have therefore

$$\bar{X}_k f = 0 , \qquad k = 1, \cdots, n-1 ,$$

on S.

We shall show first that if u is a function on N_p with

$$\bar{X}_k u = 0\,, \qquad k = 1, \cdots, n-1\,,$$

then the integral

$$G(w) = \int_{J_w} u(z_1, w)\, dz_1$$

defines a holomorphic function of w in W. We remark that each of the curves J_w is located in N_p, so the integral is well defined. To show that G is holomorphic in the variable z_k, $k > 1$, fix the remaining variables $z_j = z_j^0$, $j \neq 1, k$, and consider a closed rectifiable Jordan curve C on the Riemann surface

$$R = [z_j = z_j^0, j \neq 1, k] \cap W\,.$$

Consider the integral

$$I = \int_C dz_k \int_{J_w} u(z_1, w)\, dz_1\,.$$

If we can show that this integral vanishes for each such C, then, by Morera's theorem, G will be holomorphic in z_k.

As a closed Jordan curve in R, C has an interior, D_C. Set

$$T = F(D_C \times bD)\,,$$
$$\partial T = F(C \times bD)\,,$$
$$\alpha = u(z_1, w)\, dz_1\, dz_k\,.$$

We then obtain, by Stokes' theorem,

$$I = \int_{\partial T} \alpha = \int_T d\alpha\,.$$

Since we are integrating in the complex two-dimensional affine subspace M given by $M = [z_j = z_j^0, j \neq 1, k]$, with coordinates z_1, z_k, we can restrict ourselves to $S \cap M$, given by

$$\phi(z_1, z_2^0, \cdots, z_{k-1}^0, z_k, z_{k+1}^0, \cdots, z_n^0) = 0\,,$$

which we shall simply write as

$$\phi(z_1, z_k) = 0\,.$$

At each point of M the complex tangent space to M is spanned by the vectors $\{\partial/\partial z_1, \partial/\partial z_k\}$, and what we might call the "conjugate complex tangent space" is spanned by $\{\partial/\partial \bar{z}_1, \partial/\partial \bar{z}_k\}$. We are going to integrate over $T \subset M \cap S$. At each point of T, the rank of the Jacobian matrix (gradient)

$$(\phi_{x_1}, \phi_{y_1}, \phi_{x_k}, \phi_{y_k})$$

is one. This follows from the fact that those points of S for which $\phi_{x_1} = \phi_{y_1} = 0$ are images under F of boundary points of W (points of V'), and since $D_C \Subset W$, we must have $\phi_{x_1}^2 + \phi_{y_1}^2 > 0$ at all points of T. Therefore in a neighborhood of T, $S \cap M$ is a three-dimensional hypersurface in M.

In M the induced Cauchy-Riemann operator on $S \cap M$ is given by

$$\bar{X} = \phi_{\bar{1}} \frac{\partial}{\partial \bar{z}_k} - \phi_{\bar{k}} \frac{\partial}{\partial \bar{z}_1} .$$

It is easy to see that $\bar{X}$ is a linear combination of $\bar{X}_1$ and $\bar{X}_k$ and hence that $\bar{X}u = 0$ on T. Since $\bar{X}$ is tangent to S in $M \cap T$ and $\partial/\partial \bar{z}_1$ is not, we know that $\bar{X}$ and $\partial/\partial \bar{z}_1$ are linearly independent at each point of T, and hence span the same space as $\{\partial/\partial \bar{z}_1 , \partial/\partial \bar{z}_k\}$. Thus $\partial/\partial \bar{z}_k$ is a linear combination of $\bar{X}$ and $\partial/\partial \bar{z}_1$, given by

$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{\phi_{\bar{1}}}\left(\bar{X} + \phi_{\bar{k}} \frac{\partial}{\partial \bar{z}_1}\right) . \tag{6}$$

The relationship between the variables given by ϕ means that we can solve for either x_1 or y_1 near any point of T. If we solve for $x_1 = x_1(y_1 , x_k , y_k)$ near a point where $\phi_{x_1} \neq 0$, we have

$$\phi(x_1(y_1 , x_k , y_k), y_1 , x_k , y_k) \equiv 0$$

and hence

$$\phi_{x_1} dx_1 + \phi_{y_1} dy_1 + \phi_{x_k} dx_k + \phi_{y_k} dy_k = 0 ,$$

where dx_1 is interpreted as $d(x_1(y_1 , x_k , y_k))$ in the three-dimensional affine space with coordinates (y_1 , x_k , y_k). Using complex notation again, we obtain the relationship between the formal derivatives

$$d\bar{z}_1 = -\frac{1}{\phi_{\bar{1}}} (\phi_1 \, dz_1 + \phi_k \, dz_k + \phi_{\bar{k}} \, d\bar{z}_k) .$$

Near a point of T where $\phi_{x_1} \neq 0$ we obtain the same relationship. Using this and (6) in I, we obtain

$$\begin{aligned} I &= \int_T d(u \, dz_1 \, dz_k) \\ &= \int_T \partial(u \, dz_1 \, dz_k) + \bar{\partial}(u \, dz_1 \, dz_k) \\ &= \int_T \left[\frac{\partial}{\partial \bar{z}_1} u \, d\bar{z}_1 + \frac{\partial}{\partial \bar{z}_k} u\right] d\bar{z}_k \, dz_1 \, dz_k \\ &= \int_T \frac{1}{\phi_{\bar{1}}} (\bar{X}u) \, d\bar{z}_k \, dz_1 \, dz_k \\ &= 0 . \end{aligned}$$

We can now define the function

$$\tilde{f}(z_1 , w) = \frac{1}{2\pi i} \int_{J_w} \frac{f(\zeta, w)}{\zeta - z_1} \, d\zeta$$

which will be the extension of f from S to Q. Here (z_1, w) is a point on D_w. Fixing $z_1 = z_1^0$, we see that if we set

$$u(z_1, w;\ z_1^0) = f(z_1, w)(z_1 - z_1^0)^{-1},$$

we have $\bar{X}_j u = 0, j = 1, \cdots, n-1$, on S. By the above result $\tilde{f}$ is holomorphic in the variables of w. On the other hand, for fixed w, it is clear that $\tilde{f}$ is holomorphic with respect to z_1. Hence $\tilde{f}$ is a holomorphic function defined in Q.

It remains to show that $\tilde{f}$ takes on the boundary values f. It suffices to show that for a fixed D_w, $\tilde{f}$ takes on the values of f on J_w. Consider z_1^0 large enough so that (z_1^0, w) lies outside of $\bar{Q}$. Then letting u be the same expression as above, we may again conclude that

$$G(w;\ z_1^0) = \int_{J_w} u(\zeta, w;\ z_1^0)\, d\zeta$$

is holomorphic in w. Moreover, we can allow w to tend to the boundary V', and since u is bounded there, due to the choice of z_1^0, we see that $G(w;\ z_1^0)$ vanishes on V'. Since V' is a real hypersurface in P, we can conclude that $G(w;\ z_1^0)$ vanishes identically in W for this large value of $z_1 = z_1^0$. For $|z_1^0| > R$, for sufficiently large constant R, we have the same result, and hence

$$G(w;\ z_1^0) \equiv 0, \qquad w \in W,\ |z_1^0| > R.$$

But since $G(w;\ z_1^0)$ is holomorphic in w if z_1^0 is a point in the exterior J_w, we must have $G(w;\ z_1^0)$ identically zero there also. This implies that $\tilde{f}(z_1, w)$ takes on the boundary values f on J_w. Thus part (c) is proven. Q.E.D.

COROLLARY. *$L \neq 0$ at p implies that S is locally extendible (and hence not locally S_δ) at p.*

The following theorem is essentially a converse to the one above. Let S be a locally defined real differentiable hypersurface given by $[\phi = 0]$ as in the previous theorem.

THEOREM 3. *$L \equiv 0$ near $p \in S$ if and only if S is locally S_δ at p.*

Proof: Assume first that $L \equiv 0$ near $p \in S$. Let ν be a normal vector to S at p. Let N be a connected neighborhood of p in S small enough so that the "cylinder" N_ε obtained by translating N along ν in both directions near p by a distance $\varepsilon > 0$ is a domain containing p. We choose N small enough so that $L \equiv 0$ on N. Let B_r be a ball about p of small enough radius r so that $B_r \Subset N_\varepsilon$. Let

$$K = B_r \cap S = B_r \cap N.$$

We want to show that K is an S_δ. Let $\{\varepsilon_j\}$ be a sequence tending monotonically to zero with $\varepsilon_1 = \varepsilon$. Let $\{r_j\}$ be a sequence with $r_j > r$ so that $r_j \to r$ monotonically and $B_{r_j} \Subset N_\varepsilon$. Set $U_j = B_{r_j} \cap N_{\varepsilon_j}$. Then we see that

$$K = \bigcap_{j=1}^{\infty} U_j.$$

We have only to show that each U_j is a domain of holomorphy. It is clear that the boundary surface of U_j near a point $q \in \partial U_j$ is either locally a part of the spherical boundary of B_{r_j}, a part of one of the two translates of N by a distance ε_j, or an intersection of both of these kinds. Since $L \equiv 0$ on a translate of N, this means the complex Hessian vanishes on the complex tangent spaces there. The spherical surface of B_{r_j} is given by

$$\psi(z) = |z| - r_j = 0 ,$$

where we assume p is the origin of $\mathbb{C}^n$. Thus the complex Hessian of ψ is positive-definite on the complex tangent space at a point of ∂B_{r_j}. It then follows from the definition given in Grauert [7] that U_j is a pseudo-convex domain. The solution of the Levi problem implies that U_j is therefore a domain of holomorphy (see Grauert's paper for references). Thus S is locally S_δ at p.

Suppose $L \not\equiv 0$ near p; then we need to show that S is not locally S_δ at p. Take any ball B_r about p; then there is at least one point $q \in B_r \cap S = K$, where $L(q) \neq 0$. By Theorem 2, there is a set $Q \neq \varnothing$ depending on q and K, with $Q \cap K = \varnothing$, such that all holomorphic functions on K can be analytically continued to Q. Therefore any domain of holomorphy containing K must also contain Q, and thus we cannot find a sequence of domains of holomorphy whose intersection is K. Hence K is not an S_δ, and S is not locally S_δ at p. Q.E.D.

4. Locally Extendible Submanifolds of Higher Codimension

Using the notation introduced in Section 1, let M^k be a real k-dimensional regularly embedded differentiable submanifold of $\mathbb{C}^n$, let T_p and C_p be the real and complex tangent spaces to M^k at p, and let m be the complex dimension of C_p. It is easy to see that

$$k - n \leqq m \leqq k/2 .$$

Suppose that $m = k - n$. It then follows from standard linear algebra that there is a complex-linear change of coordinates so that

$$T_p = [y_1 = \cdots = y_{2n-k} = 0]$$

in new coordinates $z_j = x_j + iy_j$, $j = 1, \cdots, n$. M^k can be expressed as (set $l = 2n - k$)

$$\phi^i(z) = 0 , \qquad i = 1, \cdots, l , \tag{7}$$

with

$$\phi^i_{x_j}\big|_0 = 0 , \qquad j = 1, \cdots, n ,$$

$$\phi^i_{y_j}\big|_0 = 0 , \qquad j = l + 1, \cdots, n ,$$

for $i = 1, \cdots, l$. Thus the matrix

$$A = (\phi^i_{y_j}\big|_0) , \qquad i, j = 1, \cdots, l ,$$

is the nonzero part of the general Jacobian matrix evaluated at $z = 0$. Since we are assuming the submanifold to be regular, this Jacobian must be of maximal rank, l, and hence A is nonsingular. We can then solve the equations (7) for $(y_1, \cdots, y_l)$ and obtain M^k expressed in the form:

$$y_j = h^j(x_1, \cdots, x_l, z_{l+1}, \cdots, z_n), \qquad j = 1, \cdots, l, \tag{8}$$

where the functions h^j vanish to second order at $(x_1, \cdots, x_l, z_{l+1}, \cdots, z_n) = 0$.

We can consider parameters

$$(u,w) = (u_1, \cdots, u_l, w_1, \cdots, w_m) \in B \subset \mathbb{R}^l \times \mathbb{C}^m,$$

the u_i being real and the w_i complex parameters, and B a neighborhood of zero in $\mathbb{R}^l \times \mathbb{C}^m$ where the h^j are defined. By using (8) we can express M^k as a mapping

$$\psi: B \to \mathbb{C}^n$$

given by

$$u_j + ih^j(u, w) \to z_j, \qquad j = 1, \cdots, l,$$

$$w_j \to z_{j+l}, \qquad j = 1, \cdots, m.$$

In vector notation this becomes

$$(u, w) + i(h(u, w), 0) \to z.$$

Under these conditions Bishop has constructed in [4] analytic discs whose boundaries lie on M^k.

Using Bishop's idea we want to construct an analytic m-disc whose distinguished boundary lies on M^k. Let D be the unit m-disc in $\mathbb{C}^m$ and let E be its distinguished boundary as in Section 2. In order to construct an analytic m-disc whose distinguished boundary lies on M^k, it suffices to solve the following problem: Let $\tau = (e^{i\theta_1}, \cdots, e^{i\theta_m})$ and let $w(\tau)$ and $u(\tau)$ be vector-valued functions of dimensions m and l, respectively. Given $w(\tau)$ as the trace on E of a holomorphic function in D, find a real-valued continuous function $u(\tau)$ on E such that

$$u(\tau) + ih(u(\tau), w(\tau))$$

is the trace on E of a holomorphic function in D. Bishop's solution to this problem in the case where D is the unit disc in $\mathbb{C}$ and E is its boundary is carried out in [4]. This method of solution can be easily generalized to the case of analytic m-discs.

If $u(\tau)$ is a C^∞ (vector-valued) function on E, then $u(\tau)$ is the trace of a harmonic function in D (given by a Fourier series). This harmonic function is the real part of a holomorphic function f in D where $\mathcal{I}m\, f(0) = 0$. Denote the trace of $\mathcal{I}m\, f(\zeta)$ by $(Tu)(\tau)$. Suppose $u(\tau)$ is a continuous real-valued solution of the equation

$$u(\tau) = c - Th(u(\tau), w(\tau)), \tag{9}$$

where c is a given constant vector and $w(\tau)$ is given as the trace of a holomorphic function $g(\zeta)$. Then the pair

$$(u + ih(u, w), w)$$

is a mapping of E into $\mathbb{C}^n$. Moreover $u + ih(u, w)$ is the trace of a holomorphic function $f(\zeta)$ with $\mathcal{R}e\,(f(0)) = c$. The pair (f, g) is therefore a holomorphic mapping

$$(f, g): D \to \mathbb{C}^n$$

and defines an analytic m-disc whose distinguished boundary lies on M^k.

To solve (9) we define the L^2 norms on the components of u and w:

$$\|u_j\|_0 = \left(\int_E |u_j(\tau)|^2\, d\tau\right)^{1/2}, \qquad j = 1, \cdots, l,$$

$$\|w_j\|_0 = \left(\int_E |w_j(\tau)|^2\, d\tau\right)^{1/2}, \qquad j = 1, \cdots, m,$$

and, letting D^μ represent derivatives of order μ with respect to the coordinates on E, we set

$$\|u\| = \sum_{\mu=0}^{m} \sum_{j=0}^{l} \|D^\mu u_j\|_0,$$

$$\|w\| = \sum_{\mu=0}^{m} \sum_{j=0}^{m} \|D^\mu w_j\|_0,$$

$$\|(u, w)\| = \|u\| + \|w\|.$$

Define also the maximum norm over E,

$$\|u\|_m = \max_{\tau\in E} |u(\tau)|,$$

$$\|w\|_m = \max_{\tau\in E} |w(\tau)|.$$

By Sobolev's lemma (see [6], p. 232), we have

$$(10) \qquad \begin{aligned} \|u\|_m &\leqq C\,\|u\|, \\ \|w\|_m &\leqq C\,\|w\|, \end{aligned}$$

where C is a universal constant.

By using Taylor expansions of h, one can obtain Bishop's inequalities (see [4])

$$(11) \qquad \begin{aligned} \|h(u, w)\| &\leqq K\,\|(u, w)\|^2, \\ \|h(u, w) - h(\hat{u}, w)\| &\leqq K\,\|u - \hat{u}\|\,(\|u\| + \|\hat{u}\| + \|w\|), \end{aligned}$$

where $K > 1$ is a constant depending only on h and its domain of definition B, and (u, w), $(\hat{u}, w)$ are C^∞ mappings of E into B of norm $\leqq 1$. Here one must use the fact that h vanishes to second order at $(u, w) = 0$. Defining $u^0 = c$, and, by induction,

$$u^j = c - Th(u^{j-1}, w) , \qquad j = 1, 2, \cdots ,$$

for a given constant c and a given function $w(\tau)$, it is easy to prove convergence in the norm $\| \quad \|$ using the above inequalities, provided (c, w) is small enough (see [4]). We state the result in the following theorem.

THEOREM 4. *Given c and $w(\tau)$ satisfying*

$$\|(c, w)\| < (16K)^{-1} , \tag{12}$$

where $w(\tau)$ is differentiable, there exists a unique continuous function $u(\tau)$ with $\|u\| < (4K)^{-1}$ satisfying (9). *Moreover if u and $\hat{u}$ denote the solutions corresponding to the data (c, w) and $(\hat{c}, \hat{w})$, both satisfying* (12), *then*

$$\|u - \hat{u}\| \leqq \tilde{K}(\|c - \hat{c}\| + \|w - \hat{w}\| , \tag{13}$$

where $\tilde{K}$ is another constant.

The second part of the theorem is proven by a slight extension of the arguments given in Bishop's paper. This theorem then tells us that for a given set of data (c, w), where w is taken to be the trace on E of a holomorphic function in D, we obtain a set of analytic m-discs whose distinguished boundaries lie on M^k and whose size and dependence on (c, w) are known. As $(c, w) \to 0$ the analytic m-discs tend to the completely degenerate analytic m-disc at the origin.

Choose $w(\tau; r)$ to be the trace on E of the holomorphic mapping

$$(\zeta_1, \cdots, \zeta_m) \to (r\zeta_1, \cdots, r\zeta_m) ,$$

where r is a real constant $\geqq 0$. Suppose the constants $r, c_1, \cdots, c_l$ all have modulus $\leqq \varepsilon$. Then for sufficiently small ε we have $\|(c, w)\| < (16K)^{-1}$. Set

$$t = (r, c_1, \cdots, c_l)$$

and let

$$T = [0, \varepsilon] \times [-\varepsilon, \varepsilon]^l .$$

T will be a space of parameters for analytic m-discs. We can define the mapping

$$\phi_1 : T \times E \to B$$

by setting

$$\phi_1(t, \tau) = (c, w(\tau; r)) .$$

Let ϕ_2 denote the mapping which carries $\phi_1(t \times E)$ onto the solution of equation (9) given by

$$(u(c, w(t; r)), w(t; r)) ,$$

where $u(c, w)$ is the solution of (9) given by Theorem 4 corresponding to the data (c, w). Let $\phi = \phi_2 \circ \phi_1$. The mapping ψ defining M^k carries $\phi(t \times E)$, which lies in B, into M^k. Let Ψ denote this composition, $\psi \circ \phi$. Thus

$$\Psi: T \times E \to M^k \subset \mathbb{C}^n ,$$

and by the construction $\Psi(t \times E)$ is the distinguished boundary of an analytic m-disc $A(t)$. Thus there is a mapping

$$F\colon T \times \bar{D} \to \mathbb{C}^n$$

which is holomorphic on D for fixed t, and such that $F \mid T \times E = \Psi$. Set

$$A = F(T \times \bar{D}) ,$$

$$bA = F(T \times E) = \Psi(T \times E) ;$$

then A is an $(l + 1)$-parameter family of analytic m-discs in $\mathbb{C}^n$ whose family of distinguished boundaries bA lies on M^k.

LEMMA 2. *A is a continuous family of analytic m-discs containing at least one completely degenerate analytic disc $A(0)$.*

Proof: $A(0)$ is clearly the point $A = 0$. For the continuity, suppose t, $\hat{t}$ are elements of T. We have

$$\|c - \hat{c}\| + \|w - \hat{w}\| \leqq C\,|t - \hat{t}| ,$$

and hence by (10) and (13)

$$\|\phi(t, \cdot) - \phi(\hat{t}, \cdot)\|_m \leqq C\,|t - \hat{t}| ,$$

where C is a suitable constant. It follows from the differentiability of ψ that

$$\|\psi(u, w) - \psi(\hat{u}, \hat{w})\|_m \leqq C' \,\|(u, w) - (\hat{u}, \hat{w})\|_m ,$$

where C' depends only on B and ψ. These last two inequalities together imply that

$$\|\Psi(t, \cdot) - \Psi(\hat{t}, \cdot)\|_m \leqq C'' \,|t - \hat{t}|$$

for some other constant C''. Thus $bA(t)$ depends uniformly on the parameter t. Q.E.D.

Using the above construction we are able to prove our main theorem which generalizes the corollary to Theorem 2.

THEOREM 5. *Suppose $p \in M^k$, $k > n$, and M^k is generic at p with $m(p) = k - n$. If $L \neq 0$ at p, then M^k is locally extendible at p.*

Remark. Bishop [4] proves that under stronger conditions on $M^4 \subset \mathbb{C}^3$ one can prove local extendibility, where the extension set is open as in the case of a hypersurface in Theorem 2 and in Lewy's example [13]. Examples similar to that of Lewy show that under the above assumptions one can not necessarily extend to an open set.

Proof: Let $X_1, \cdots, X_m$ be complex vector fields on M^k which span the complex tangent spaces to M^k near p (here $m = k - n$). Let $l = 2n - k$ and let

$Y_1, \cdots, Y_l$ be l real vector fields on M^k so that $\{\mathscr{R}e\, X_1, \cdots, \mathscr{R}e\, X_m, \mathscr{I}m\, X_1, \cdots, \mathscr{I}m\, X_m, Y_1, \cdots, Y_l\}$ span the real tangent spaces to M^k near p. The set $\{X_1, \cdots, X_m\}$ is complex integrable (cf. [14], [16]) if

$$[X_i, \bar{X}_j] = \sum_{j=1}^{m} (a_j X_j + b_j \bar{X}_j),$$

where $\{a_j\}$ and $\{b_j\}$, are C^∞ functions on M^k near p.

By hypothesis, $L(v) \neq 0$ at p for some $v \in C_p$. Let X be a vector field whose restriction to p is v. Then

$$X = \sum_{j=1}^{m} \alpha_j X_j$$

and

$$[X, \bar{X}] = \sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j [X_i, \bar{X}_j].$$

In general, each commutator is a vector field on M^k near p and hence is a linear combination of the spanning vector fields. Thus we have

$$[X_i, \bar{X}_j] = \sum_{\mu=1}^{m} (b_\mu(i,j) X_\mu + c_\mu(i,j) \bar{X}_\mu) + \sum_{\nu=1}^{l} d_\nu(i,j) Y_\nu,$$

so that

$$\pi[X, \bar{X}]_p = \sum_{\nu=1}^{l} \sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j\, d_\nu(i,j) Y_\nu,$$

where the functions $d_\nu(i,j)$ are evaluated at p. Since $\pi[X, \bar{X}]_p \neq 0$, at least one of the coefficients $d_\nu(i,j)$ must be different from zero at p and hence near p. This implies that the set $\{X_1, \cdots, X_m\}$ is not complex integrable near p. This in turn implies that M^k does not contain any complex submanifolds of complex dimension m near p since along such a complex submanifold the set $\{X_1, \cdots, X_m\}$ would be complex integrable (see [16]).

We want to show now that M^k is locally extendible. Let N_0 be a neighborhood of p in M^k in which there are no complex submanifolds of complex dimension m and let N be any smaller such neighborhood of p. Let A be the continuous family of analytic m-discs given by Lemma 2, where ε is chosen small enough so that bA is contained in N. Set $Q(t) = A(t) - M^k$, and let

$$Q = \bigcup_{t \in T} Q(t).$$

Consider a fixed $t \in T$ such that $r > 0$. For such a t, $A(t)$ has as a projection on the subspace of $\mathbb{C}^n$ spanned by the variables $\{z_{l+1}, \cdots, z_n\}$ the m-disc of radius r,

$$\{(z_{l+1}, \cdots, z_n) : |z_j| < r, j = l+1, \cdots, n\},$$

and hence $A(t)$ is a nondegenerate analytic m-disc of m-complex dimensions. $A(t) \cap M^k$ contains at most complex submanifolds of dimension $m - 1$, so $Q(t)$

must contain a two-dimensional subset. Hence Q has two-dimensional subsets and does not intersect M^k. Moreover we claim that the mapping

$$H(N \cup \bar{Q}) \to H(N)$$

is onto. If $f \in H(N)$, then $f \in H(bA)$ and f is defined in a domain $B \supset N$. By Lemma 2 and Theorem 1, there is a domain $U \supset B$ with $U \supset A$ and a holomorphic function $g \in H(U)$ such that $g \mid B = f$. Thus the above mapping is onto, and N is extendible to $N \cup \bar{Q}$. Hence M^k is locally extendible at p. Q.E.D.

COROLLARY. *Suppose $p \in M^k$, $k > n$, and $m(p) = k - n$. If M^k contains no complex submanifolds of complex dimension m near p, then M^k is locally extendible at p.*

Remark 1. In the proof of the above theorem, A was constructed as an $(l + 1)$-parameter family of nondegenerate analytic m-discs. A is therefore the image of a mapping of a real $(k + 1)$-dimensional cell $T \times \bar{D}$ into $\mathbb{C}^n$. It is conjectured that this mapping is differentiable and that the image of $T \times \bar{D}$ under this mapping is essentially a $(k + 1)$-dimensional submanifold of $\mathbb{C}^n$ with boundary, with perhaps singularities on lower dimensional subsets, and that $bA \cap M^k$ is a submanifold of M^k passing through p. If this were so, one could ask for the conditions under which one could or could not extend to higher dimensions from this new submanifold. We hope to return to this question later.

Remark 2. One might also expect that an analogue of Theorem 2 were true. Examples seem to indicate this, but at present this problem remains unsolved.

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